

GENERALIZED MODELS AND BENCHMARKS  
FOR CHANNEL COORDINATION

A Dissertation  
by  
AYŞEGÜL TOPTAL

Submitted to the Office of Graduate Studies of  
Texas A&M University  
in partial fulfillment of the requirements for the degree of  
DOCTOR OF PHILOSOPHY

August 2003

Major Subject: Industrial Engineering

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# ABSTRACT

Generalized Models and Benchmarks

for Channel Coordination. (August 2003)

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This dissertation takes into account the latest industrial trends in integrated logistical management and focuses on recent supply chain initiatives enabling the coordination of supply chain entities. The specific initiatives of interest rely on carefully designed transportation and supply contracts such as Vendor Managed Inventory applications. With such new initiatives, substantial savings are realizable by carefully coordinating the operational decisions, such as procurement, transportation, inventory, and production decisions, for different cooperating entities in the supply chain. The impact is particularly tangible when coordinated policies address channel coordination issues between these entities.

This dissertation first provides a critical review and comparative analysis of the literature on buyer-vendor coordination problems. Recognizing a need for analytical research in the field, the dissertation then develops and solves centralized and decentralized models for complex buyer-vendor coordination problems with applications in supply/replenishment and transportation/delivery contract design. The two specific classes of problems considered include i) buyer-vendor coordination under generalized replenishment costs, and ii) buyer-vendor coordination under depreciating economic value of items. Under these considerations, the dissertation also develops efficient coordination algorithms and new mechanisms for effective channel coordination.

To my Parents and my Sister

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## CHAPTER I

### INTRODUCTION

The buyer-vendor coordination problem is one of the classical research areas in the multi-echelon inventory literature. Within the large spectrum of existing work in this area, *centralized* and *decentralized* models can be considered as the two extremes. Classical multi-echelon inventory theory suggests integrating and modeling the decision problems of the vendor and the buyer together. This strategy qualifies as centralized modeling of the problem and, without any doubt, provides the best result in terms of total system cost (i.e., the global optimum). However, it requires that both the vendor and the buyer make their data available to the other party. In application, this may not be desirable unless both parties represent components of the same company. Furthermore, in real life, there is often a superior/subordinate relationship inherent in the situation where the dominant party prefers her/his priorities to lead the solution. As a result, decentralized modeling of the problem may be necessary. In a decentralized model, the parties solve their subproblems independently of each other with very limited sharing of information. As a consequence, the superior party's priorities lead the solution. In most retail applications, the superior party is the buyer whereas in manufacturing applications the superior party is usually the manufacturer, i.e., vendor.

While classical buyer-vendor coordination models can generally be characterized as falling into one of the two extreme modeling approaches (i.e. centralized vs. decentralized), the current trend in supply chain research is towards investigating ways to apply decentralized models without sacrificing too many of the cost saving benefits

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This dissertation follows the style and format of *Management Science*.

that result from centralized models. For this purpose, *channel coordination* tries to identify the inefficiencies in decentralized solutions and to align the individual incentives for both parties with those of the centralized solutions. The output of channel coordination, i.e., the so-called *coordinated solution*, combines the benefits of the two extremes, i.e., centralized and decentralized solutions. However, the pure centralized and the pure decentralized modeling approaches are still important techniques for practical applications because the former sets a benchmark for cost minimization whereas the latter helps to identify opportunities for coordination.

In a buyer-vendor setting, a centralized solution is particularly useful for Vendor Managed Inventory (VMI) practices (Çetinkaya and Lee 2000, 2002). VMI is a supply chain initiative where the vendor is authorized to manage inventories of agreed-upon stock-keeping units at downstream locations, e.g., retailers. In particular, after using advanced data retrieval and information sharing systems to review the inventory at the retailer, the vendor makes decisions regarding the timing and quantity of delivery for re-supply. Not surprisingly, VMI has gained more attention as information technology has advanced and the cost of information sharing has decreased. In some VMI applications, the vendor not only manages the inventory at the retailer but also owns it; Proctor & Gamble and WalMart, for example, use this practice. Information sharing and centralized modeling approaches for buyer-vendor coordination in such settings are crucially important for the performance of the system as a whole. While the benefits of a centralized modeling approach are inherent in the motivations of strategic alliances such as VMI, supply chain entities are not always linked by collaborative relationships. Enhanced by businesses on the internet, these entities often act in a decentralized and competitive manner to increase their profits.

Because both centralized and decentralized models have important practical applications, an important question is how the solutions of the two models differ from

each other and under what circumstances their implications are similar. Although similar questions have been studied in the production and inventory literature since the early 1970's (e.g., Goyal 1976, Lee and Rosenblatt 1986, Schwarz 1973), the answers to these questions have recently become more important for obtaining coordinated solutions and formalizing these solutions in contractual agreements, e.g., a VMI contract or a transportation/delivery contract.

In keeping with the recent trends in supply chain practice, the goals of this dissertation are:

- To identify optimal coordinated policies for integrated transportation and inventory decisions in buyer-vendor systems.
- To develop a modeling framework and theoretical understanding of channel coordination issues in the context of new initiatives in supply chain management.
- To address the question of under what conditions coordination works and to render insights into contract design and operational level decision making.

Ultimately, we hope to demonstrate how centralized solutions can either be used by supply chain entities who fully coordinate their operational planning or as benchmarks to identify the inefficiencies in decentralized solutions and thereby obtain solutions for effective channel coordination.

## **I.1. Scope of the Dissertation**

In order to achieve these goals, this dissertation will analyze both deterministic and stochastic demand problems with the following primary objectives:

- Provide a critical review and comparative analysis of the literature on buyer-vendor coordination problems.

- Develop and solve centralized and decentralized models for complex buyer-vendor coordination problems with applications in supply/replenishment and transportation/delivery contract design, and
- Develop practical coordinated solutions for these problems for effective channel coordination.

More specifically, we investigate the following problems.

### **I.1.1. Buyer-Vendor Problem with Generalized Replenishment Costs**

As we discuss in detail in Chapter II, the existing literature in buyer-vendor coordination generally overlooks important transportation considerations. In particular, the impact of cargo capacity constraints and generalized inbound/outbound transportation cost functions are rarely taken into account in previous work. However, we know that substantial system-wide cost efficiencies may be achievable by carefully incorporating such transportation considerations with inventory replenishment decisions in buyer-vendor systems. With this motivation, in Chapters III, IV, V and VI, we consider a replenishment cost structure of the form

$$C(Q) = K + \lceil Q/P \rceil R \quad (1.1)$$

where the first term (i.e.  $K$ ) is a fixed cost and the second term is the total truck cost in proportion to the number of trucks used. Here  $P$  is the truck capacity;  $R$  is the per truck cost; and  $Q$  is the replenishment quantity. Under the assumptions of this generalized replenishment cost structure, we study the buyer-vendor coordination problem in four different settings as we discuss next.

#### **I.1.1.1. Integrated Pure Inventory Problem with Deterministic and Constant Demand**

We generalize the deterministic demand buyer-vendor coordination problem to consider cargo capacity constraints and general inbound/outbound transportation costs, simultaneously. That is, we analyze the effects of the generalized cost structure  $C(Q)$  given by Equation (1.1) on the inventory decisions of a buyer and his/her vendor using a centralized approach. Modeling the interaction between the two parties in this manner is particularly important in VMI systems. In these systems, typically, it is the vendor who incurs the transportation costs as well as the inventory holding costs both at the upper and lower echelons, i.e., vendor's and buyer's warehouses, respectively.

First, we consider the case where no production occurs at the vendor, and this is why we call this model the “Pure Inventory Model.” The vendor orders in bulk and dispatches to the buyer in smaller quantities. Hence, the number of buyer replenishments within one replenishment cycle of the vendor is an integer variable. Due to the stepwise structure of the replenishment cost function and the integer decision variables, it is theoretically challenging to minimize total cost functions in these types of systems.

Under the above assumptions, in Chapter III we study two specific problems using the centralized modeling approach. The dissertation first develops and solves the simpler problem where the general replenishment cost structure is modeled for the vendor only. Then the model is extended to the case where both the vendor and the buyer are subject to this replenishment cost structure. Hence, in the second model, the inbound transportation costs/constraints are modeled explicitly both for the buyer and the vendor. For each case, heuristic algorithms with error bound

analyses are provided. Using the costs of these heuristics as upper bounds, finite-time exact solution procedures are also developed.

#### **I.1.1.2. Channel Coordination for the Pure Inventory Problem with Deterministic and Constant Demand**

The centralized solutions of the class of buyer-vendor problems studied in Chapter III can be used in two ways: 1) For coordinating the operational planning activities of the buyer-vendor pair under full cooperation. 2) As a benchmark to identify inefficiencies in the decentralized solutions of the buyer and the vendor individually in order to obtain solutions for effective channel coordination. Hence, in Chapter IV, we utilize these centralized solutions as benchmarks in the following way. First, we discuss how to compute the decentralized solutions for the buyer and the vendor individually. Then, on a large set of problem instances, we compare the cost efficiency of decentralized and centralized modeling. We report the parameter values and problem characteristics under which the solutions of the two modeling approaches are close to, or far apart from, each other. We then propose ways for the vendor and the buyer to coordinate their decentralized decisions to obtain improved solutions in terms of costs.

#### **I.1.1.3. Integrated Production-Inventory Problems with Deterministic and Constant Demand**

While some of the recent literature investigates channel coordination issues in buyer-vendor systems, another stream of research concentrates on centralized production-inventory optimization problems in complex settings. In keeping with these research trends, we also study the case where the vendor is a manufacturer and has a finite production rate. Hence, we refer to such models as “Production-Inventory Models.”

In the integrated production-inventory problem, the pattern of the vendor's dispatch policy to the buyer significantly effects the profits of the system. In Chapter V, we discuss seven major dispatch policies that have been proposed in the previous literature. Some of these policies exhibit easily implementable structures; others are more cost effective but have very complex structures. We propose a unified model for the integrated buyer-vendor production-inventory problem that can be reduced to the previously developed models for the seven major dispatch policies identified in the literature. Using this unified model, the seven policies are tested against the optimal policy through a careful numerical study. As a result of this study, we propose insights for a buyer and a vendor to use in selecting the dispatch policy that best fits their system in terms of structure, robustness and costs.

As a final task in this area, in Chapter V we incorporate the general replenishment cost structure  $C(Q)$ , given by Expression (1.1), into the integrated buyer-vendor production-inventory problem under different dispatch policies. It turns out that for some of the dispatch policies, the corresponding problems are solvable using the algorithms we developed in Chapter III.

#### **I.1.1.4. Single Period Stochastic Demand Channel Coordination Problem with Generalized Replenishment Costs**

In Chapter VI, we consider the buyer-vendor problem where the buyer operates in a Newsboy environment. More specifically, we consider the case where the buyer faces a single-period stochastic demand, and the problem is to compute the centralized and decentralized order quantities under the generalized replenishment cost structure given by  $C(Q)$  in Expression (1.1). We also address channel coordination issues in this context.

The Newsboy setting is important for two reasons. First, it is widely applicable



for items with short product life cycles, such as consumer electronics, software products, fashion goods, etc. Secondly, solving the Newsboy Problem provides the solution for the last period of the corresponding multi-period stochastic problem. There are also some interesting properties of the Newsboy setting that are different from the infinite horizon setting we consider in Chapters III, IV, V. Specifically, the motivation to order more products, so that economies of scale can be achieved, is restricted to a single period. That is, in contrast to the EOQ-type models with infinite planning horizons, increasing the quantity of goods purchased or produced does not result in savings from fixed costs in future periods.

We first model and solve the Newsboy setting problem with generalized replenishment costs under the single-period expected profit maximization objective using centralized and decentralized approaches. We present several interesting properties of the expected profit functions which simplify the solution methodology. We then propose novel coordination mechanisms by which the buyer and the vendor can coordinate their decentralized decisions.

A common result of the coordination problems in Chapter IV and Chapter VI is that, contrary to common belief in the literature, it is not always better to motivate the buyer to order more in order to coordinate the channel. When the vendor has a replenishment cost structure of the form  $C(Q)$ , there are opportunities for the vendor and the buyer to take advantage of full truck loads to share the benefits. In certain cases, this can make smaller order quantities from the buyer more advantageous. The coordination mechanisms proposed take these cases into account.

### **I.1.2. Buyer-Vendor Problem under Depreciating Economic Value of Items**

The current literature on buyer-vendor coordination assumes that the retail price, that is the selling price of items at the buyer, is a constant or is a function of the order

quantity. However, especially with items that have short product life cycles, a natural decline occurs in their economic value over time. For example, most retail stores selling seasonal items successively discount their prices during the season because of this depreciation in economic value. These kinds of markdowns are referred to as “permanent markdowns” in the marketing literature. In Chapter VII, we consider two specific problem groups that involve permanent markdowns as summarized below.

#### **I.1.2.1. Single Period, Single Replenishment Model with Time Dependent Retail Price**

In our first model, we analyze a single period stochastic buyer-vendor problem where the buyer successively decreases his/her selling price during the planning horizon. We assume that this decrease in selling price is due to permanent markdowns and that there is no prior advertisement. Therefore, customers come to the store at normal rates. We also assume that customers have different preferences in their willingness to pay. That is, some customers prefer to buy the item earlier in the season even if it is expensive while others buy it towards the end of the season when it is cheaper. As a result, the demand arrival process is modeled as a pure Poisson Process. In Chapter VII, we consider the profit maximization problem for this setting using both decentralized and centralized modeling approaches. Based on a comparative analysis of these approaches, we illustrate that time is an important component of the pricing strategy and that an efficient coordination mechanism should take the length of the planning horizon into consideration.

### **I.1.2.2. Single Period, Single Replenishment Model with Time Dependent Demand and Retail Price**

In our second model, we consider the case where demand depends on the retail price. In this case, we assume that demand arrivals form a pure Non-homogenous Poisson Process where the rate is a function of the selling price which itself depends on time. We present centralized and decentralized models for this problem in Section VII.2.

## **I.2. Organization of the Dissertation**

The dissertation is organized as follows. Chapter II provides a review of the current literature on the buyer-vendor coordination problem. In Chapter III, we study centralized inventory/transportation models with infinite horizon, deterministic and constant demand assumptions. In Chapter IV, we address the issue of channel coordination for the problems in Chapter III. The production models and their extensions are analyzed in Chapter V. Chapter VI presents an analysis of single period stochastic demand problems with generalized transportation costs. This is followed by Chapter VII which discusses coordination in cases where items have depreciating economic value.

## CHAPTER II

### LITERATURE REVIEW

In this chapter, we present a critical review of the buyer-vendor coordination literature that emphasizes transportation, production/inventory and channel coordination issues. The buyer-vendor coordination problem forms the basis of multi-echelon inventory theory since it considers only two stock-keeping locations. On the other hand, early works in the multi-echelon inventory literature, such as Clark and Scarf's (1960) seminal paper on serial systems with stochastic demand, concentrate on more general systems. These authors develop a periodic review inventory control model and derive the optimal ordering policy for a single-product, serial system facing independent, identically distributed demand. Using Clark and Scarf's (1960) work as a base, many of the researchers that followed them study the problem of finding the optimal timing for, and the optimal quantity of, material flows in serial systems (e.g., Federgruen and Zipkin 1984a, 1984b, Debo and Graves 1985, Rosling 1989). Federgruen (1993) provides a review of these studies. Starting in the early 90's, interest in multi-echelon inventory theory grew as a result of new industry practices that improved coordination between different stock-keeping locations, as well as various theoretical challenges, to form the basis for what we now call *supply chain management*.

Our definition of supply chain management is a broad one that includes all of multi-echelon inventory theory and more. Classical works in multi-echelon inventory theory assume that the entities (i.e., stock-keeping locations) cooperate and hence solve their problems using a centralized approach. As discussed in Chapter I, this is a valid assumption if the entities belong to the same company or operate under long-term agreements such as VMI systems. Supply chain management, on the other hand, considers both logistical and informational issues as well as dominance relationships

between the entities. Consequently, supply chain studies focus on more than just the system-wide optimization of inventory problems.

Current research in inventory management places an increased emphasis on the integration/coordination of different functional specialties within a firm as well as throughout a supply-chain. Rather than using classical centralized modeling to coordinate these entities, the focus of most recent studies is on channel coordination. In this dissertation, we present both centralized and decentralized models for complex supply/replenishment, transportation/delivery problems with the goal of obtaining coordinated decentralized solutions for these problems. Therefore, we will limit this review to work that is closely related to the theme of this dissertation. For an excellent discussion of the fundamental results that emerged from the 60's through the 90's, we suggest Muckstadt and Roundy's (1993) review of multistage production/inventory models.

In our review, we will summarize the related buyer-vendor coordination literature under three main headings: 1) Transportation Considerations in the Buyer-Vendor Coordination Literature, 2) Production/Inventory Models, and 3) Channel Coordination Models. Before going into the details of our analysis, we provide a more technical description of channel coordination and a discussion of the importance of transportation costs for channel coordination which we believe will help to better relate our work to the literature and to highlight its contributions. That is, we discuss the basics of channel coordination in Section II.1. Next, in Section II.2, we provide a summary of Goyal's (1976) buyer-vendor coordination problem, which is referred to as the classical problem or model throughout the dissertation. We proceed with a discussion of the literature in Section II.3 and conclude with a summary in Section II.4.

## II.1. Channel Coordination

“Channel coordination” is a phrase coined in the marketing literature that applies to improving the total expected system profits in a decentralized model and to bringing them closer to those of a centralized model (see Tsay et al. 2000). Letting the optimum values of the total expected system profits in the centralized and decentralized models be  $\pi^c$  and  $\pi^d$ , respectively, their values can be found using the solutions to the following two problems:

$\mathcal{CM}$  : Centralized Model

$$\begin{aligned} \max \quad & E[(BUYER's PROFITS + VENDOR's PROFITS)] \\ \text{s.t.} \quad & Buyer's constraints \\ & Vendor's constraints \end{aligned}$$

$\mathcal{DM}$  : Decentralized Model

$$\begin{aligned} 1) \mathcal{BP} : \quad & \max \quad E[(BUYER's PROFITS)] \\ & \text{s.t.} \quad Buyer's constraints \\ \\ 2) \mathcal{VP} : \quad & \max \quad E[(VENDOR's PROFITS)] \\ & \text{s.t.} \quad Buyer's solution output \\ & \quad \quad Vendor's constraints \end{aligned}$$

The objective function value of the first problem gives the value of  $\pi^c$ . Furthermore, we define  $\pi_b^c$  and  $\pi_v^c$  as the expected values of the buyer and the vendor profits resulting from the solution of the  $\mathcal{CM}$ . Accordingly,  $\pi^c = \pi_b^c + \pi_v^c$ .

In the decentralized problem, the subproblems are solved sequentially. Whether the buyer or the vendor solves his/her subproblem first depends on which party dominates the system. The formulation given in  $\mathcal{DM}$  belongs to a “buyer driven channel” where it is the buyer who has greater dominance. The first subproblem (i.e.

$\mathcal{BP}$ ) is executed by the buyer, and its solution gives the expected value of the buyer's profits in the decentralized model (i.e.  $\pi_b^d$ ). Similarly, the objective function value of the second subproblem (i.e.  $\mathcal{VP}$ ) is the vendor's expected profits in the decentralized model (i.e.  $\pi_v^d$ ). It follows that  $\pi^d = \pi_b^d + \pi_v^d$ .

Since the centralized model maximizes the expected system profits, its objective function value is an upper bound on the total expected profits of the buyer-vendor system. Hence, in any system,  $\pi^d \leq \pi^c$ . In this sense, the centralized model can be used as a benchmark, and the gap between  $\pi^d$  and  $\pi^c$  can be considered as an inducement to improve the decentralized solution. In the following list, we summarize some important observations from our analysis of the models introduced above.

- $\pi^d \leq \pi^c$  : The decentralized solution is inferior to the centralized solution as far as system profits are concerned.
- $\pi_b^d \geq \pi_b^c$  : The buyer's expected profits in the decentralized solution are at least as great as those in the centralized solution.
- $\pi_v^c \geq \pi_v^d$  : The vendor's expected profits in the centralized solution are at least as great as those in the decentralized solution.
- $\pi_v^c - \pi_v^d \geq \pi_b^d - \pi_b^c$  : The vendor's gain from the centralized solution is no less than the buyer's loss from the decentralized solution.

We note that the above observations are based on the decentralized model formulation we presented above. If a “vendor driven channel” is assumed (i.e. vendor solves his/her subproblem first), then the second and third inequalities are true for the vendor and the buyer, respectively; and the fourth inequality changes direction. Under the “buyer driven channel” assumption, the second observation follows because in the decentralized model, the buyer's feasible region for his/her decision variables

attains its maximum. The first and second observations, together with the fact that  $\pi^c = \pi_b^c + \pi_v^c$  and  $\pi^d = \pi_b^d + \pi_v^d$ , imply the third and fourth observations.

The fourth observation is the key to the idea of channel coordination because it suggests that one party's gain from the centralized solution is greater than the other party's loss. That is, the vendor's gain from using the centralized solution can be used to compensate the buyer's relative losses under the centralized solution as well as to increase the vendor's profits under the decentralized solution. This requires that the decentralized solution be coordinated in such a way that it results in the same outcome for the decision variables as does the centralized solution and also a mutually agreeable way of sharing the resulting profits. The sharing can be done by means of fixed payments between the parties, quantity discounts, rebates, etc., or a combination of these. It can be negotiable between the parties or implicitly forced by one party to influence the behavior of the other. All of these ways for achieving centralized profits using a decentralized approach are called *coordination mechanisms*.

Now consider a buyer-vendor system where the decision variable is the buyer's order quantity. As will be discussed in more detail in subsequent parts of the current chapter, the major theme of the classical literature is that it is always better for the vendor to encourage the buyer to increase his/her order quantity. The following proposition generalizes this idea to any setting where the vendor's expected profits in the optimization problem can be written in terms of the buyer's order quantity.

**PROPOSITION 1** When the vendor's expected profits are an increasing function of the buyer's order quantity, the buyer's optimal order quantity in the centralized model is no less than his/her optimal order quantity in the decentralized model.

**Proof:** Let  $\Pi_v(Q)$  be the vendor's expected profit function in terms of the buyer's order quantity. Similarly let  $\Pi_b(Q)$  be the buyer's expected profits resulting from an



order quantity of  $Q$  units. Denote the optimum levels of the buyer's order quantity as an outcome of the centralized and decentralized solutions by  $Q_c^*$  and  $Q_d^*$ . Therefore,  $\pi_b^c = \Pi_b(Q_c^*)$ ,  $\pi_v^c = \Pi_v(Q_c^*)$ ,  $\pi_b^d = \Pi_b(Q_d^*)$  and  $\pi_v^d = \Pi_v(Q_d^*)$ . We have  $\pi^d \leq \pi^c$ , and, therefore,  $\pi_b^d + \pi_v^d \leq \pi_b^c + \pi_v^c$ . Since  $\pi_b^d \geq \pi_b^c$ , it follows that  $\pi_v^c - \pi_v^d \geq 0$  and hence  $\Pi_v(Q_c^*) \geq \Pi_v(Q_d^*)$ . When the vendor's expected profits are an increasing function of the buyer's order quantity, this implies that  $Q_c^* \geq Q_d^*$ . ■

Although the above proposition sounds fairly comprehensive, there are many practical cases where the vendor's expected profits do not increase with the buyer's order quantity. One such practical situation is when the vendor has a generalized replenishment cost structure such as in Expression (1.1). We show in Chapters IV and VI that with this cost structure for the vendor, there are cases where  $Q_c^* < Q_d^*$ , and we propose new coordination mechanisms that consider this case.

To this end, it is important to note that although “channel coordination” is a new term for the operations research literature, the concept is not a new one. The benefits of centralized modeling over decentralized modeling in terms of system costs was first illustrated by Goyal (1976). Based on this idea, Monahan (1984) was the first to use discounts as a means for the vendor to encourage the buyer to order more and, hence, to increase his/her own profits without changing the buyer's cost. Although the term “channel coordination” was not used in this early literature, we believe that the ideas and the models proposed became the basis for subsequent studies in the area. Therefore, we review them in Section II.3.3 on channel coordination models. Goyal and Gupta (1989) provide an excellent survey of this early literature on buyer-vendor coordination.

Goyal's (1976) work is also the foundation for many of the models in this dissertation. In the later sections, we refer to his case as the “classical buyer-vendor problem” which we describe next.

## II.2. The Classical Buyer-Vendor Problem

Goyal considers a relatively simple situation where the demand rate at the buyer is a known constant,  $D$ . Both the vendor and the buyer operate in a classical EOQ-like setting. That is, the vendor replenishes his/her inventory from an outside supplier with ample supply and sends replenishments to the buyer as needed. Both the vendor and the buyer replenishments are delivered instantaneously, and each incurs fixed costs. The time between two successive random vendor replenishments is called the vendor replenishment cycle whereas the time between two successive buyer replenishments is called the buyer replenishment cycle. Since the vendor orders in bulk and replenishes the buyer periodically in smaller, equal-sized dispatches during an infinite planning horizon, we have  $Q_v = nQ_b$  where  $Q_v$  denotes the vendor's replenishment quantity;  $n$  is the number of buyer replenishments in a vendor replenishment cycle and  $Q_b$  denotes the buyer's replenishment quantity. The problem is to find the optimal values of  $Q_v$  and  $Q_b$  in order to meet the cost minimization objective, where fixed replenishment as well as inventory holding costs for both the vendor and buyer, are considered explicitly. Using the above relationships, this problem is equivalent to computing the optimal values of  $Q_v$  and  $n$ .

Figure 1 provides an illustration of the inventory load profiles of the buyer and the vendor when  $n = 4$ . Here  $T_v$  and  $T_b$  represent the vendor's and buyer's replenishment cycle lengths, respectively. Since demand is a known constant, in order to avoid lost-sales or backorders, we have  $Q_b = DT_b$  and  $Q_v = DT_v$  so that  $Q_v = DnT_b$ .

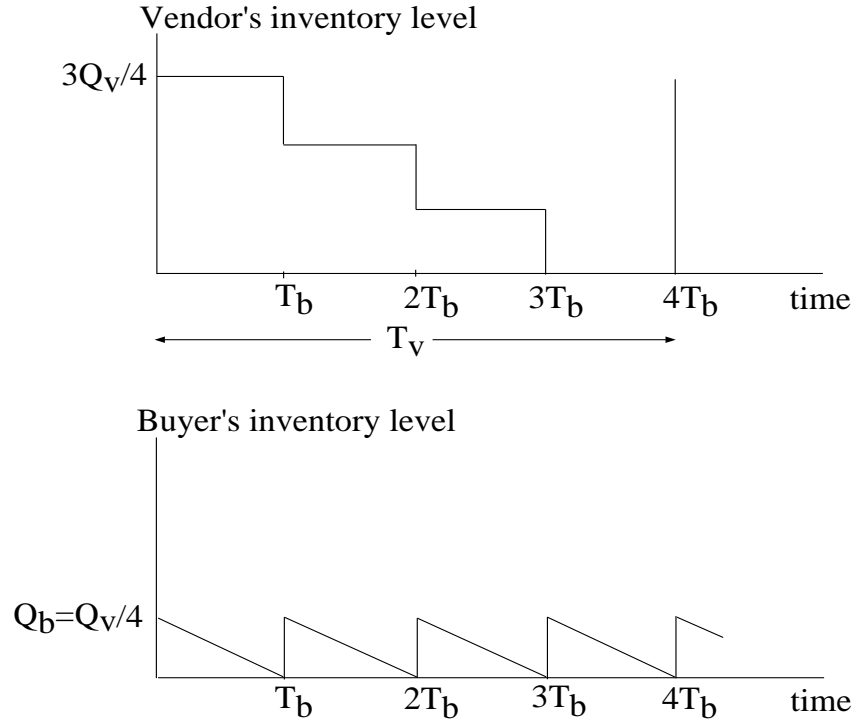
The original paper by Goyal (1976) models the problem using both centralized and decentralized approaches. However, it is the centralized approach that is visited as the basic deterministic model in multi-echelon inventory systems in many of the

books on inventory theory (e.g., Silver et al. 1998, pages 477–481). The total system-wide costs per unit time in the centralized approach is given by

$$\frac{DK_b}{Q_b} + \frac{h_b Q_b}{2} + \frac{K_v D}{nQ_b} + \frac{h_v(n-1)Q_b}{2}$$

where  $K_b$  and  $K_v$  are the buyer's and the vendor's fixed replenishment costs, and  $h_b$  and  $h_v$  are their inventory holding costs/unit/time, respectively. The first two terms of the above expression are the buyer's total costs per unit time, and the last two terms are the vendor's total costs per unit time. The solutions of the centralized and decentralized models of this problem are further discussed in Chapter IV.

Figure 1 Inventory Load Profiles in the Classical Buyer-Vendor Problem



## II.3. A Detailed Analysis of the Literature

### II.3.1. Transportation Considerations in Buyer-Vendor Coordination Literature

In examining the previous work in buyer-vendor coordination, we found that, with the exception of a few papers (Chan et al. 2002, Hoque and Goyal 2000, Çetinkaya and Lee 2002), the previous work in buyer-vendor coordination ignores practical transportation considerations when developing optimization models. Therefore, analyzing the impact of generalized transportation costs on buyer-vendor coordination problems is one of the main contributions of this dissertation.

Chan et al. (2002) model a single-warehouse multi-retailer problem, and they incorporate piece-wise linear cost structures representing common-carrier transportation charges. They analyze the case where the warehouse does not hold any inventory, i.e., acts as a cross docking point. On the other hand, Hoque and Goyal (2000) model a single-vendor single-buyer problem where the vendor's replenishment/production rate is finite. Their model incorporates a capacity constraint limiting the replenishment quantities of the buyer. This capacity constraint may be interpreted as the cargo capacity of the *outbound* transport device, e.g., truck or cargo capacity, and the underlying formulation implicitly assumes that a single transportation source, e.g., a single truck, is available. In another study by Çetinkaya and Lee (2002), the authors again model and solve a single-vendor single buyer, deterministic demand problem where the replenishment cost structure  $C(Q)$  in Expression (1.1), is considered only for the outbound transportation of the vendor and the buyer does not hold any inventory. One common characteristic of these three earlier works is that they concentrate on the case of deterministic demand where the vendor replenishes from an outside supplier (i.e., the vendor has an infinite replenishment rate). The scope of

this dissertation is more general than these previous papers in the following sense:

- We concentrate on the case of a single vendor, single buyer (a.k.a the one-warehouse one-retailer problem), but we allow inventory at both locations, i.e., we explicitly consider the case of two stock-keeping locations.
- We study the case where, if necessary, more than one truck can be utilized both for inbound and outbound transportation, and we model a generalized cost structure representing this opportunity.
- We analyze both deterministic and stochastic demand problems. For the case of deterministic demand, we study both infinite and finite replenishment rate problems.

It is worth noting that the impact of practical transportation considerations, and, in particular, general transportation cost structures, has been investigated in the context of single-echelon inventory lot-sizing models. This body of research is closely related to our work since it provides a foundation for lot-size optimization in a simpler setting. For previous work addressing single echelon lot-sizing models with general transportation assumptions, the reader is referred to Aucamp (1982), Lee (1986), Lee (1989), Russell and Krajewski (1991). Aucamp (1982) treats a modification of the standard economic order quantity (EOQ) model in which the total inbound freight cost is partially determined by the integer number of carloads/trucks required to fill the order. Russell and Krajewski (1991) also consider the standard EOQ problem, and they model a transportation cost structure for less-than-truckload (LTL) shipments reflecting reductions in freight rates when the replenishment quantity exceeds one of the nominal rate breakpoints. Lee (1986, 1989) extends Aucamp's work (Aucamp 1982) to consider a more general freight cost structure with quantity discounts and

dynamic demands, respectively. In fact, the optimization technique developed by Lee (1986) for the single echelon model provides a foundation for the buyer-vendor system considered in Chapter III of this dissertation.

It is important to note that the existing work with transportation considerations (e.g., Chan et al. 2002, Hoque and Goyal 2000, Çetinkaya and Lee 2002) use the centralized modeling approach. The channel coordination problem in the context of transportation costs has not yet been investigated. Another observation is that, no study considers both the inbound and outbound transportation capacities and costs for the vendor or their effects on the replenishment decisions of the buyer and the vendor.

As discussed earlier, the issue of centralized vs. decentralized modeling approaches was first raised by Goyal (1976). After that, many other researchers worked on the same problem under different settings and even proposed some simple ways to coordinate the channel (e.g., Monahan 1984, Banerjee 1986b, Lee and Rosenblatt 1986). In the meantime, another stream of research developed on buyer-vendor coordination problems. A group of researchers worked on the problem of finding the best dispatching policy for minimizing the total replenishment and inventory related costs in a buyer-vendor system where the vendor is a manufacturer with a finite production rate. In the next section, we summarize these studies.

### **II.3.2. Production/Inventory Models**

An important generalization of the classical buyer-vendor problem is when the vendor is a manufacturer and has a finite production rate. We can conclude from the existing literature that the type of the dispatch policy used to deliver buyer replenishments is particularly important in this setting. This is because production at the vendor may continue while the buyer is being replenished, and according to the type of the

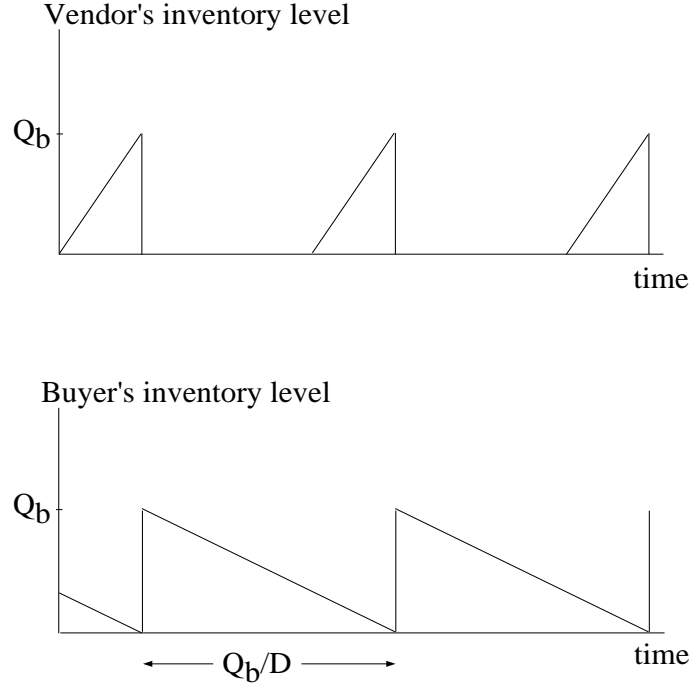
dispatch policy, this may result in very complex inventory load profiles for both the buyer and the vendor. This in turn effects the inventory holding costs.

The optimal dispatch policy for this problem was identified by Hill (1999), and it has a very complex structure that may be difficult to implement in practice. On the other hand, there are some simpler policies proposed in the literature, but they are not always optimal. The earliest and the simplest of these policies is the “Lot-for-Lot” (LFL) Policy (Banerjee 1986b). With this policy, the vendor’s production lot size and the buyer’s order size are equal and the same from cycle to cycle (see Figure 2). Banerjee (1986b) also compares individual decision making and joint optimization and proposes a method for sharing the benefits of the latter by making price adjustments. Goyal (1988) extends this study by changing the replenishment policy. That is, the vendor is allowed to produce one large lot to supply an integer number of orders of the purchaser. If this integer is one, the cost is identical to that in Banerjee (1986). One restriction of this study is that the vendor is assumed to finish production of the whole batch before releasing the first shipment to the buyer.

Another dispatch policy proposed in the literature is the “Identical Delivery Quantity,” (IDQ) Policy (Banerjee and Burton 1994, Lu 1995). With this policy, equal-sized dispatches are made to the buyer as in the Lot-for-Lot Policy, but the production lot size of the vendor and the dispatch sizes are not necessarily equal. Figure 3 shows the inventory load profiles of a single buyer and a single vendor in the case of the IDQ Policy.

The IDQ Policy was first introduced by Banerjee and Burton (1994) who consider a single vendor and multiple buyers. They first model the problem using the decentralized approach. They assume that the vendor’s inventory depletes at a constant rate, which is the sum of the demand rates of the buyers. Each buyer individually optimizes his/her costs using the EOQ assumptions. Then the vendor’s total cost

Figure 2 Inventory Load Profiles for the Lot-for-Lot Policy

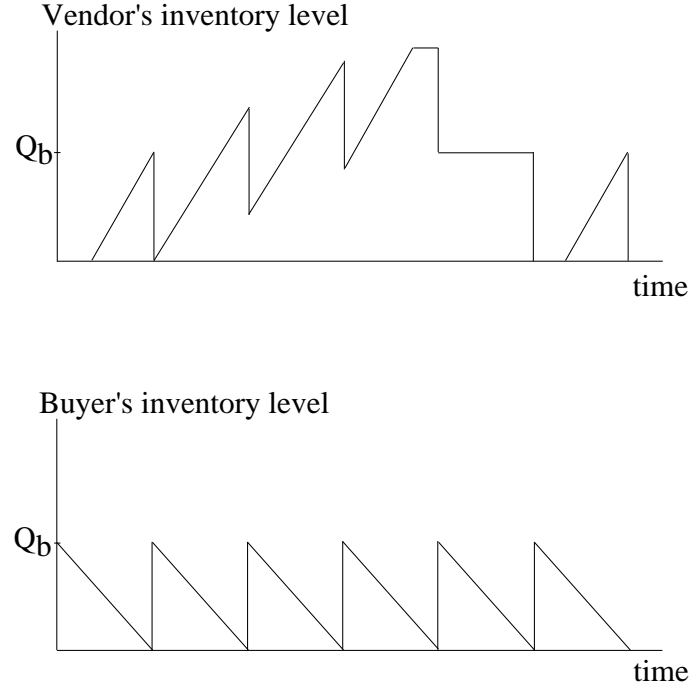


is represented in terms of the buyers' optimal order sizes and is optimized for the vendor's replenishment lot size. The authors also test their model to see the effects of discrete inventory depletions using simulation. Then, they model the same problem using a centralized approach. In order to coordinate the independent buyers, and for the sake of analytic tractability, they assume that the vendor's replenishment cycle time is an integer ( $K$ ) multiple of the buyers' ordering cycle time which is common to all buyers. For a set of numerical examples, the authors illustrate that the proposed centralized model results in lower system costs than the decentralized model.

Note that both Goyal (1988) and Banerjee and Burton (1994) assume that the production lot size for the vendor is an integer multiple of the ordering lot size of the purchaser. In this sense, the single buyer case for Banerjee and Burton (1994)



Figure 3 Inventory Load Profiles for the IDQ Policy



is similar to Goyal (1998). However, in Goyal (1988), the vendor does not replenish the buyer unless he finishes the production of the whole lot whereas in Banerjee and Burton (1994), the vendor is allowed to replenish the buyers during production, i.e., before the lot is completed.

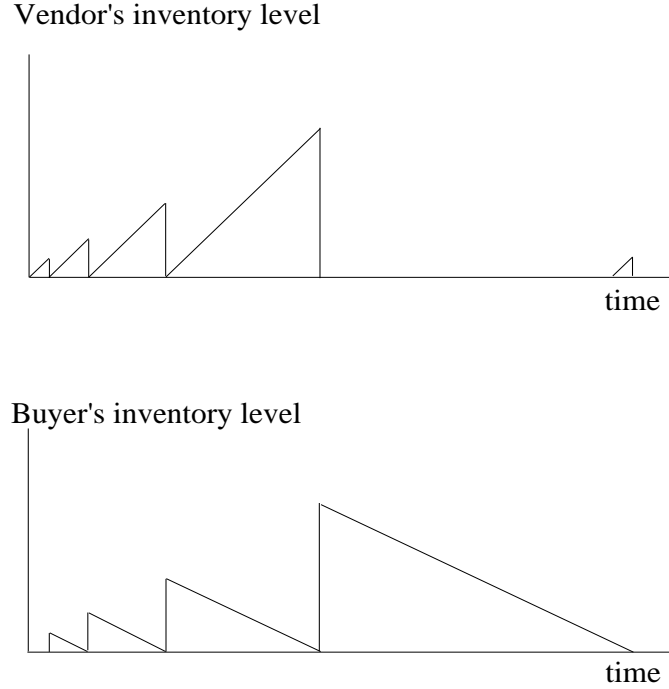
The IDQ Policy is also studied in a paper by Lu (1995) for a single vendor, multi-buyer setting. The author proposes a decentralized model where the vendor decides on the replenishment quantities by minimizing his/her total average cost, subject to the maximum percentage that each buyer can deviate from his/her optimal cost. The ordering cycle times of the buyers are given by  $K_i T$  where  $T$  is the replenishment cycle time of the vendor and  $K_i \in \{1, 2, 3, \dots\} \cup \{1/2, 1/3, \dots\}$ . Unlike in Banerjee and Burton (1994), where the ordering cycle times of all the buyers are the same,

here the buyers' cycle times are different. But again, for a single buyer, the policy can be characterized as IDQ. A heuristic approach is provided for the multi-buyer case, and the problem is solved optimally for the single buyer. Note that, in the single buyer problem, if the constraint regarding the maximum cost that the buyer can incur is ignored, then Lu's model (Lu 1995) reduces to the one in Banerjee and Burton (1994).

It is worth noting that in the studies mentioned until now, none of the authors focus on improving the system costs by changing the dispatch policy. Banerjee (1986b) and Banerjee and Burton (1994) aim to analyze the impact of centralized and decentralized decision making on the individual and joint costs. Lu (1995) argues that, by putting constraints on the buyers' maximum costs, a vendor can pass some of his/her savings on to the buyers in order to decide on replenishment quantities agreeable by all parties in a vendor driven channel. While the earlier literature on production/inventory models can be characterized in this way, later the attention of the researchers diverts to finding the optimal dispatching policy in this kind of a setting. One of the policies that has been proposed in an effort to improve total system costs is the "Deliver What is Produced" (DWP) Policy (Goyal 1995). Figure 4 shows the inventory load profiles of a buyer and a vendor in a case of the DWP Policy.

The DWP Policy was introduced into the literature by Goyal (1995). The studies summarized above all assume that the replenishment quantity for any buyer is the same in each dispatch. However, Goyal (1995) relaxes this assumption for the single vendor and single buyer case. He assumes that  $q_{i+1} = (\vartheta/D)q_i$  where  $q_i$  is the size of the  $i^{th}$  shipment to the buyer and  $\vartheta$  and  $D$  are the vendor's production rate and the buyer's demand rate, respectively. The system cost in this policy can be represented as a function of the size of the first shipment to the buyer and an integer, denoted by  $n$ , representing the number of buyer replenishments within a production cycle of the

Figure 4 Inventory Load Profiles for the DWP Policy



vendor. A solution procedure that evaluates all possible  $n$  and compares the optimal costs for different values of  $n$  is proposed. However, this solution procedure is not necessarily finite.

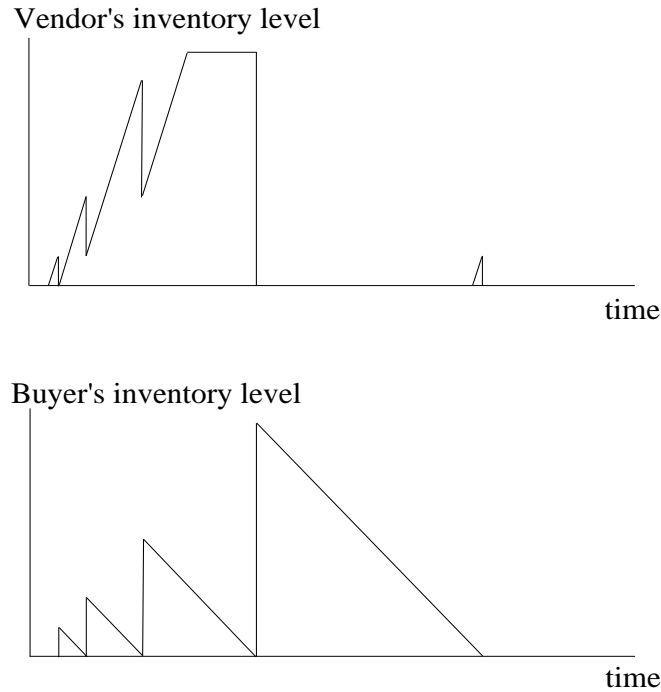
A study by Viswanathan (1998) compares the DWP policy with the IDQ policy for the single vendor, single buyer problem. After an extensive numerical analysis, Viswanathan (1998) concludes that no policy is better than the other for all parameter values. As the buyer's unit inventory holding cost increases, IDQ becomes a better policy, and as the demand rate increases with respect to production rate, DWP is superior to IDQ.

The idea of different sized shipment quantities to the buyer is also adapted by Hill (1997), who proposes a more generalized model for Goyal's problem (Goyal 1995).

He assumes that the size of the  $(i + 1)^{st}$  shipment can differ from the size of the  $i^{th}$  shipment by a factor of  $\lambda$  which can take any value in the range  $[1, \vartheta/D]$ . In the later parts of the dissertation, we refer to this policy as the “Factor- $\lambda$ ” (F $\lambda$ ) Policy (see Figure 5 for the inventory load profiles). The total cost as a result of this policy is represented as a function of the increase factor ( $\lambda$ ), the number of buyer replenishments within a production cycle of the vendor ( $n$ ) and the size of the first shipment to the buyer ( $q_1$ ). The solution procedure proposed by Hill (1997) makes a full search over  $\lambda$  within  $[1, \vartheta/D]$  and  $n$ . A range of values that limits the possible values of  $n$  is also proposed based on a numerical analysis. However, this range does not necessarily include the optimal solution. Later, Goyal (2000) improves the policy proposed by Hill (1997) by modifying the shipment quantities. We refer to the resulting dispatch policy in Goyal (2000) as the “Improved-Factor- $\lambda$ ” (IF $\lambda$ ) Policy.

Another policy for the single vendor, single buyer production/inventory problem is proposed by Goyal and Nebebe (2000). In this policy, the last  $n - 1$  shipments to the buyer are of equal size, given by  $\vartheta/D$  times the initial shipment size (see Figure 6). Therefore, we call this policy the “1-unequal” Policy. The average annual system cost can be written as a function of the first shipment size and the number of dispatches within one vendor production cycle (i.e.,  $n$ ). An analytical solution is provided for the optimum values of the decision variables. Some problem instances are also noted where the current policy performs better than Goyal (1995), Lu (1995), and Hill (1997).

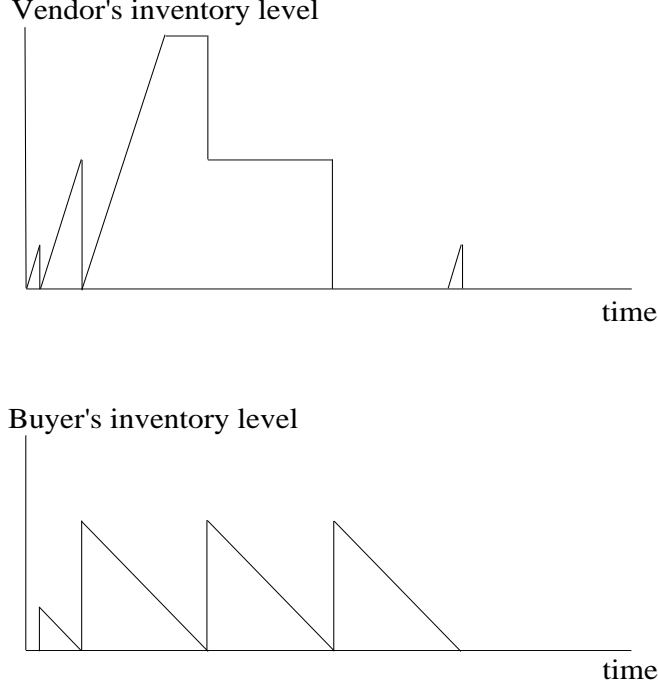
In a recent study, Hoque and Goyal (2000) generalize the 1-unequal Policy by assuming that the first  $e$  shipments to the buyer are of unequal size and increase by a factor of  $\vartheta/D$  after which they stay constant for the remaining  $n - e$  shipments. Unlike in the previous models, Hoque and Goyal (2000) also incorporate a capacity constraint which sets an upper limit on the maximum allowable dispatch size. If

Figure 5 Inventory Load Profiles for the Factor- $\lambda$  Policy

this maximum value is large enough so that the constraint is not binding, then the model proposed solves the integrated inventory replenishment problem of the buyer and vendor under a new dispatch policy. We refer to this policy as the “e-unequal” Policy. Figure 7 shows the inventory load profiles of a buyer and a vendor under this policy.

All the production/inventory models that have been covered so far assume a certain dispatch policy and then find the optimum parameter values for that policy. The global optimum for the single buyer, single vendor production problem has been found by Hill (1999). It turns out that the structure of the optimal sequence of shipments varies according to problem parameters and contains all of the above patterns as special cases. However, one common property is that the first period’s shipment

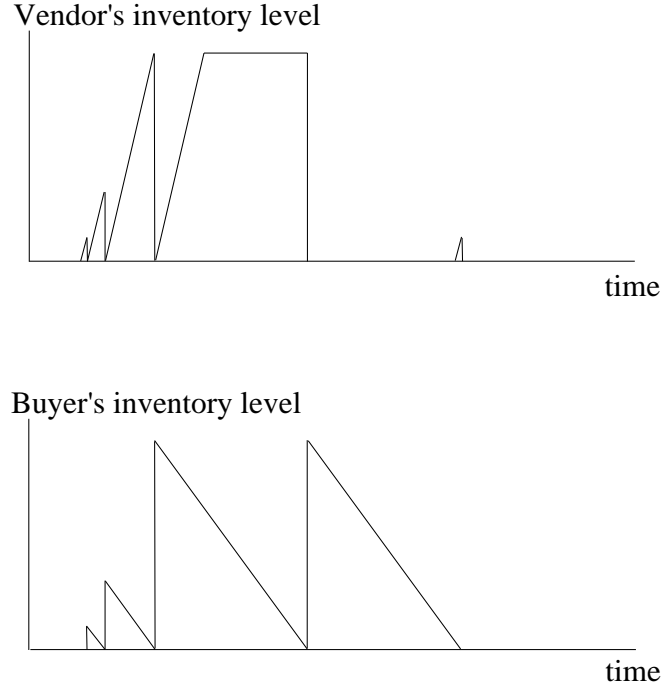
Figure 6 Inventory Load Profiles for the 1-unequal Policy



quantity is the least and the others are nondecreasing. For example; one possible solution for the sequence of shipments may be  $q, \lambda q, \lambda q, \dots, \lambda q$ . This is similar to the structure that Goyal and Nebebe (2000) assume except that  $\lambda$  is not necessarily equal to  $\vartheta/D$ . If  $\lambda = 1$ , this results in equal sized shipments as in Banerjee and Burton (1994) and Lu (1995). As another possibility, the sequence of shipments may follow the pattern:  $q_1, (\vartheta/D)q_1, (\vartheta/D)^2q_1, \dots, (\vartheta/D)^{m-1}q_1, q_{m+1} = q_{m+2} = \dots = q_n$ . This structure is similar to that assumed by Hoque and Goyal (2000), except for the fact that the equal shipment sizes are not necessarily equal to the size of the last unequal shipment. As a final possibility, it may turn out that  $q_{i+1} = (\vartheta/D)q_i$  as is the case in Goyal (1995).

In Table I, we summarize the production/inventory models for two-echelon sys-

Figure 7 Inventory Load Profiles for the e-unequal Policy



tems. The fourth column in Table I states the modeling approach used in each paper. For those papers that model their problem using a decentralized approach, we also state whether they assume a buyer-driven or vendor-driven channel, as discussed in Section II.1. Note that for the centralized modeling approach, this attribute is not applicable.

Another stream of research considers multiple production stages where the problem is to find optimal production lot sizes and transfer batch sizes from one stage to the next. The production lot size is assumed to be uniform across all stages, and only a single set-up cost is allowed for uninterrupted production at each stage. However, the number of transfer batch sizes at each stage may differ. Under these assumptions, Goyal and Szendrovits (1986) model the problem for equal and unequal sized

Table I Summary of Literature on Production/Inventory Models

Author	No. of Items	Replenishment Policy	Focus	Buyer vs. Vendor Driven Channel
Banerjee (1986b)	Single	Lot-for-Lot	Centralized vs. Decentralized	Buyer
Banerjee and Burton (1994)	Multi	IDQ	Centralized vs. Decentralized	Buyer
Lu (1995)	Multi	IDQ	Decentralized	Vendor
Goyal (1995)	Single	DWP	Centralized	NA
Viswanathan (1998)	Single	IDQ and DWP	Centralized IDQ vs. DWP	NA
Hill (1997)	Single	Factor- $\lambda$	Centralized	NA
Goyal (2000)	Single	Improved-Factor- $\lambda$	Centralized	NA
Goyal and Nebebe (2000)	Single	1-unequal	Centralized	NA
Hoque and Goyal (2000)	Single	e-unequal	Centralized	NA

NA: Not Applicable

batch shipments between production stages. The size of the largest unequal batch is taken as the batch size for equal shipments. A heuristic algorithm is provided, but its performance is not adequately tested (i.e., it is not compared with the optimal solution).

For the same problem, Hoque and Kingsman (1995) provide an exact optimization method. In a recent study, Bogaschewsky et al. (2001) solve the problem assuming unequal transfer batches. These authors report cases that outperform the results of Goyal and Szendrovitz (1986).

### II.3.3. Channel Coordination Models

As discussed earlier, the channel coordination concept was first introduced in the marketing literature. Although the specific term was coined later, the idea of channel coordination was captured earlier by many researchers who proposed quantity discounts as a mechanism to influence the buyer's ordering pattern. Dolan (1987),



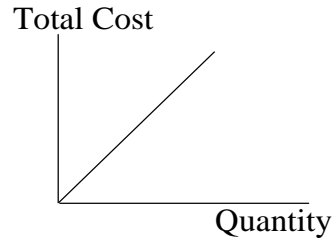
in his survey paper, lists another function of quantity discounts as the achievement of price discrimination against a single customer or a set of heterogeneous customers (i.e., customers with different cost and revenue parameters). Since price discounts are commonly used as a mechanism for channel coordination, we review below some price schedules that include different forms of quantity discounts.

In a *uniform price schedule*, there is a linear, through the origin, relationship between the quantity and the total price (see Figure 8.a). Another class of price schedules is the *nonlinear price schedule* which has three forms: 1) Two-part tariff schedule, 2) Two-block tariff schedule (or Incremental discount schedule), 3) All units discount schedule. A *two-part tariff schedule* imposes a fixed charge for any goods purchased and a uniform price for each unit (see Figure 8.b). In a *two-block tariff*, a.k.a. incremental discount schedule, a per unit price  $p_1$  is charged for any unit, up to quantity  $x$ , at which point the per-unit price changes to  $p_2$  for all units greater than  $x$  (see Figure 8.c). Another common price schedule is the *all units discount* in which a lowered price applies to all units if a certain quantity level is exceeded (see Figure 8.d). These are the price schedules that appear most commonly in the literature. A detailed discussion of different price schedules, with examples from real life cases, is provided in Dolan (1987).

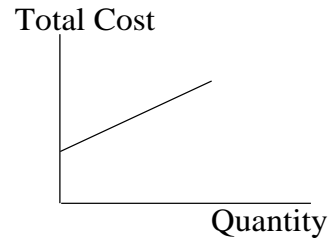
Another commonly cited work from the marketing literature is by Jeuland and Shugan (1983). The authors propose joint ownership, simple contracts, implicit understandings, profit sharing and quantity discounts as mechanisms for achieving channel coordination. In their models, they assume that the demand at the buyer is a decreasing function of retail price. They propose price schedules for channel coordination in a single buyer, single vendor system as well as in a single vendor, multiple buyers system. However, as they also specify in the paper, the price schedule they propose for the multiple buyers case violates the Robinson Patman Act. This act de-

Figure 8 Price Schedule Illustrations

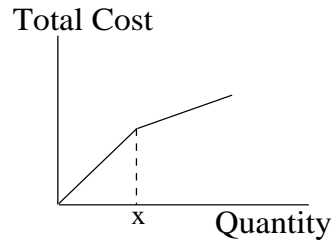
(a) Uniform Price Schedule



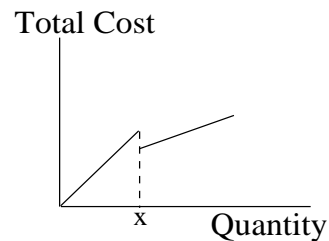
(b) Two-Part Tariff Schedule



(c) Two-Block Tariff Schedule



(d) All Units Discount Schedule



clares it illegal for a vendor to give different terms to different buyers. In fact, in the case of multiple heterogeneous buyers, coordinating the channel in compliance with the Robinson Patman Act is a challenging research problem. As will be discussed in the sequel, some later researchers have worked on this problem (Lal and Staelin 1984, Hoffman 2000).

While the marketing literature studies by Dolan (1987) and Jeuland and Shugan (1983) are good examples of the state of the work done in the channel coordination area in the early 1980's, we cite Monahan (1984) and Lal and Staelin (1984) as the two pioneering papers in the operations research literature.

Monahan (1984) studies a single vendor, single buyer problem where the vendor's replenishment lot size is equal to the buyer's order quantity per cycle. The buyer's

replenishment problem is modeled using the EOQ Model; however, the inventory holding costs of the vendor are not incorporated into the model. The author addresses the case where the vendor may ask the buyer to increase his/her ordering size by a factor (i.e.,  $k$ ). The paper does not assume any ordering relationship between the magnitudes of the buyer's and the vendor's fixed replenishment costs. It turns out that the optimal value of  $k$  (i.e.,  $k^*$ ), under the assumptions of the Monahan's paper, is given by  $\sqrt{1 + K_v/K_b}$ , where  $K_v$  and  $K_b$  are the fixed replenishment costs of the vendor and the buyer, respectively. This implies that, regardless of the values of the setup costs,  $k^*$  is always greater than one. Hence, the vendor is always better off encouraging the buyer to increase his/her order size. In fact, one can conclude this result directly by using Proposition 1, which we proved in Section II.1. Changing his/her ordering behavior results in increased costs for the buyer. However, Monahan (1984) proposes an all-units discount schedule with one break point through which the parties can share the benefits of coordination. As a result of the specific price schedule proposed, the buyer changes his/her order quantity while staying in a "no worse profit" situation.

In a later study, Banerjee (1986a) modifies the model of Monahan (1984) to incorporate the inventory holding costs of the vendor. The vendor in this study is a manufacturer with a finite production rate. The author shows that, in this setting, there may be cases where smaller order quantities from the buyer are better for increasing channel profits. However, it is pointed out that this theoretical result is rare in practice. Nevertheless, the optimal value of the  $k$  factor found by Banerjee (1986a) does not only depend on the ratios of ordering costs but also on the unit inventory holding costs.

Another generalization of Monahan (1984) is further examined by Lee and Rosenblatt (1986). They point out that Monahan (1984) does not put any constraint on

the maximum amount of the discount. Hence, in some cases Monahan’s result could imply that the discount amount exceeds the vendor’s unit price for the item. Considering a unit profit margin (i.e., the minimum amount of profit that the vendor aims for per item) the authors incorporate a constraint on the model that limits the maximum value of the unit discount. They also assume that the replenishment lot size of the vendor is an integer multiple of the replenishment lot size of the buyer. They follow the exact methodology of Monahan (1984); however they take  $k \geq 1$ , apriori. Recall that Monahan (1984) does not include this condition in his model, but it is a logical extension of his assumptions. Goyal (1987) improves the solution procedure developed by Lee and Rosenblatt.

As we stated earlier, Lal and Staelin (1984) published one of the major studies in the early operations research literature on channel coordination. In this study, the authors investigate the channel coordination problem for the single vendor, multiple buyers case. They first analyze the problem assuming that the buyers are homogenous (i.e. they have the same cost and revenue parameters). This problem is equivalent to the single buyer, single vendor case. This is because the price schedule that is optimal for one buyer applies to all the others. In subsequent parts of the paper, they also analyze the heterogenous buyers’ case. In both cases, they propose a price schedule that is continuous but changes at certain intervals. For example, for a system with homogenous buyers, the unit price is constant up to the centralized order quantity; then it decreases exponentially. For an optimal price schedule, they find the parameters of this exponential decrease analytically, and it guarantees that the buyer stays in a “no worse” situation. In the heterogenous buyers case, they again restrict the pricing strategy so as to be continuous. The optimal price schedule is computed via an algorithmic approach, and the resulting price schedule does not violate the Robinson Patman Act. However, it only guarantees that the buyers order their individual

optimal quantities (i.e., decentralized order quantities). This approach does not necessarily put all of the buyers in a “no worse” situation. Sometimes the vendor may have to give more incentives to some of the buyers to discourage them from taking advantage of the price regions in the price schedule that are specifically designed for other buyers.

The channel coordination problem for a system with heterogeneous buyers is also studied by Hoffman (2000). However, unlike Lal and Staelin (1984), he restricts the price schedule to an all-units discount schedule with multiple breakpoints and considers discounted cash flow as the performance measure of the buyers and the vendor. Neither the vendor nor the buyers have inventory holding costs; they just incur fixed replenishment costs. The parameters of an optimal price schedule is again found using an algorithmic approach. However, as in Lal and Staelin (1984), in the resulting price schedule, the vendor may be required to give more discounts to some buyers than those required in order to put them in a “no worse” situation.

An interesting generalization of the deterministic fixed demand rate problem is the case where the demand rate is a function of the selling price at the buyer. It is demonstrated by Weng (1995) that in this case (i.e., demand is a decreasing function of selling price), channel coordination cannot be guaranteed by quantity discounts alone. This is because the unit selling price at the buyer is a decision variable under the control of the buyer, and the buyer may choose a unit selling price that maximizes his profits. Thus, a fixed payment (franchise fee) paid by the buyer to the supplier may also be required.

The studies that we have reviewed so far assume that demand is deterministic. The deterministic models help us to gain insights into the dynamics of the problem. Also, their solutions can be used as approximations for time-varying and stochastic demand problems. However, stochastic models provide better representations of real

life applications. On the other hand, the number of studies on channel coordination that investigate the stochastic nature of the problem is very limited. One such study was conducted by Parlar and Weng (1997). The authors analyze the coordination problem between the manufacturing and the supply departments of a firm. The setting is similar to that of the Newsboy Problem with the exception that if demand is not met with the production quantity of the first run, then a second cycle is initiated. Since the costs associated with this second run are higher, the supply department orders an additional quantity of the material as a precaution against excess demand. However, as demand is random, a stockout situation can still occur. The authors analyze both the integrated and the independent decision making models of the buyer and the vendor. They provide conditions for which the former model is significantly better than the other. However, they do not provide any mechanisms for channel coordination.

In two recent studies, Lau and Lau (2003), and Ertek and Griffin (2002) investigate different aspects of the channel coordination problem. In the first study, the authors show that specific results on channel coordination under the price dependent demand assumption cannot be generalized. Utilizing different demand curves as functions of retail price, they illustrate the effects of price-demand relationships on channel coordination. In a second study, Ertek and Griffin (2002) compare vendor-driven versus buyer-driven channels as a solution to the decentralized problem. Although these authors analyze a very simple setting, the impact of the power structure is an interesting issue. To our best knowledge, the whole literature on channel coordination assumes a buyer-driven channel. Therefore, we believe this paper provides the groundwork for designing vendor-driven channels.

In Table II, we provide a summary of literature on channel coordination.

Table II Summary of Literature on Channel Coordination

Author	Demand	Planning Horizon	Upstream Party	Buyer's Costs	Vendor's Costs	Retail Price	Demand & Retail Price Dependency	No. of Items
Jeuland and Shugan (1983)	Det.*	One-time purchasing	Warehouse	Fixed set-up	Fixed set-up	Constant (dec. variable)	$\frac{\partial Demand}{\partial Retail Price} < 0$	Multi
Monahan (1984)	Det.	Infinite	Warehouse	Fixed set-up+ Inv. Holding	Fixed set-up	Constant (parameter)	No dependency	Single
Lal and Staelin (1984)	Det.	Infinite	Warehouse	Fixed set-up+ Inv. Holding	Fixed set-up Inv. Holding	Constant (parameter)	No dependency	Multi
Banerjee (1986a)	Det.	Infinite	Manuf.'er	Fixed set-up+ Inv. Holding	Fixed set-up+ Inv. Holding	Constant (parameter)	No dependency	Single
Lee and Rosenblatt (1986)	Det.	Infinite	Warehouse	Fixed set-up+ Inv. Holding	Fixed set-up Inv. Holding	Constant (parameter)	No dependency	Single
Weng (1995)	Det.	Infinite	Warehouse or Manuf.'er	Fixed set-up+ Inv. Holding	Fixed set-up+ Inv. Holding	Constant (dec. variable)	$\frac{\partial Demand}{\partial Retail Price} < 0$	Single
Parlar and Weng (1997)	Stoch.**	Finite	Warehouse	Fixed set-up+ Backorder	Fixed set-up	Constant (parameter)	No dependency	Single
Hoffman (2000)	Det.	Infinite	Warehouse	Fixed set-up	Fixed set-up	Constant (parameter)	No dependency	Multi
Lau and Lau (2003)	Det.	One-time purchasing	Warehouse	Purchasing	Purchasing	Constant (dec. variable)	$\frac{\partial Demand}{\partial Retail Price} < 0$	Single
Ertek and Griffin (2002)	Det.	Infinite	Warehouse	Fixed set-up+ Inv. Holding	Purchasing	Constant (parameter)	No dependency	Single

\*: Deterministic, \*\*: Stochastic

## II.4. Summary

In this chapter, we have reviewed the literature on buyer-vendor coordination studies that emphasizes transportation models, production/inventory models and channel coordination models. As we stated in the introduction to this chapter, the classical multi-echelon inventory models can also be considered as part of the buyer-vendor literature under the condition that the parties fully cooperate. However, since there is a vast body of literature on multi-echelon systems, we have only reviewed some representative papers in this area. It is important to note that the case with multiple buyers (see Roundy 1985, 1986) is an important generalization of the single buyer, single vendor problem within the context of multi-echelon inventory theory.

Another notable generalization of the problem is when demand is dynamically changing or stochastic. To our best knowledge, there is no study in the literature that considers the channel coordination problem under a dynamically changing demand assumption. However, we cite Arkin et al. (1989), Chan (2002), Diaby and Martel (1993), Joneja (1990) for dynamic demand problems in multi-echelon inventory theory. It is also worthwhile to note that the existing research on the dynamic demand case concentrates on the computational challenges associated with the development of cost efficient replenishment policies. Although the multi-echelon inventory literature contains many studies based on the stochastic demand assumption, such studies are very limited in the channel coordination literature (Parlar and Weng 1997, Lee et al. 2000).

Based on the general characteristics of the existing literature discussed above, we list some further observations:

- While there are many studies that expand on Monahan (1984), several important generalizations, such as the deterministic multi-item problem and deter-



ministic problems with general transportation and supply/replenishment costs, are still open research areas.

- As mentioned earlier, in contrast to the classical models, Weng (1995) considers the case where the selling price at the buyer is a decision variable and the demand rate is deterministic but is a function of the selling price. The stochastic demand version of this problem is an open research area.
- It appears from the current literature that the more complex channel coordination problems in stochastic environments are largely untreated. These include
  - stochastic demand problems with general transportation and supply/ replenishment costs,
  - stochastic demand problems with procurement cost/price dependent demand, and
  - multi-item stochastic demand problems.

To achieve channel coordination, the buyer-vendor pair can ideally come together and decide between themselves to operate at the centralized order quantity. However, this is less feasible in practice. In reality, the determination of how they split the benefits depends on the relative bargaining power of the parties. Another alternative is for the vendor to send the buyer a price schedule that is a function of order quantity. This way, he/she can use prior information about the buyer's reactions to develop a price schedule that will influence the buyer's ordering behavior. We believe that this is a more practical way of coordinating the channel, and, therefore, in the later parts of the dissertation, we adopt this strategy.

## CHAPTER III

INTEGRATED PURE INVENTORY PROBLEM WITH DETERMINISTIC AND  
CONSTANT DEMAND

In this chapter, we extend the earlier work in buyer-vendor coordination with deterministic and constant demand in order to consider: i) the cargo capacity constraints for both *inbound* and *outbound* transport equipment, and ii) the general transportation cost structure  $C(Q)$  that may represent a fleet of vehicles, rather than a single truck, explicitly.

The assumptions of the problem considered here are essentially the same as those for the classical buyer-vendor coordination model (Goyal 1976), with the exception that our case takes into account a general replenishment cost structure, which includes a fixed replenishment/delivery cost as well as a stepwise truck/cargo cost for both the vendor and the buyer. The vendor's replenishment cost is

$$\mathcal{C}_v(Q_v) = K_v + \left\lceil \frac{Q_v}{P} \right\rceil R, \quad (3.1)$$

where  $Q_v$  denotes the vendor's replenishment quantity;  $K_v$  denotes the fixed replenishment cost;  $P$  denotes the truck/cargo capacity; and  $R$  denotes the truck/cargo cost. As a result, the vendor's replenishment cost includes both a fixed portion  $K_v$  and a freight cost that is proportional to the number of trucks/cargoes used. A cost of  $R$  per truck/cargo is incurred whether it is fully loaded or partially loaded. Similarly, the buyer's replenishment cost is given by

$$\mathcal{C}_b(Q_b) = K_b + \left\lceil \frac{Q_b}{P} \right\rceil R, \quad (3.2)$$

where  $Q_b$  denotes the buyer's replenishment quantity, and  $K_b$  denotes the fixed replenishment cost at the buyer. Note that cost functions  $\mathcal{C}_v(\cdot)$  and  $\mathcal{C}_b(\cdot)$  include the vendor's

inbound and outbound cargo costs represented by the stepwise terms  $\lceil Q_v/P \rceil R$  and  $\lceil Q_b/P \rceil R$ , respectively. We consider the case where the vendor or vendor's transportation contractor, such as a third party logistics company, manages both the inbound and outbound transportation. Hence, the same type of transportation equipment is used with a similar truck/cargo cost  $R$  and capacity  $P$ . We also take into account the inventory holding costs per-unit per-unit-time at the vendor, denoted by  $h_v$ , and at the buyer, denoted by  $h_b$ . The echelon holding cost is denoted by  $h' = h_b - h_v \geq 0$ .

We develop two models. **Model I** is a special case where the inbound cargo costs are modeled explicitly but the outbound cargo costs are ignored. **Model II** is a generalization of Model I, and it considers both the inbound and outbound cargo costs.

A summary of the notation and mathematical formulations for Models I and II are given in the next section. Model I is analyzed in Section III.2 whereas Model II is analyzed in Section III.3. For both models, we develop heuristic solution procedures with error bound analysis. The costs of the heuristic solutions are used as upper bounds in obtaining finite time exact solution procedures. The chapter concludes with a summary in Section III.4.

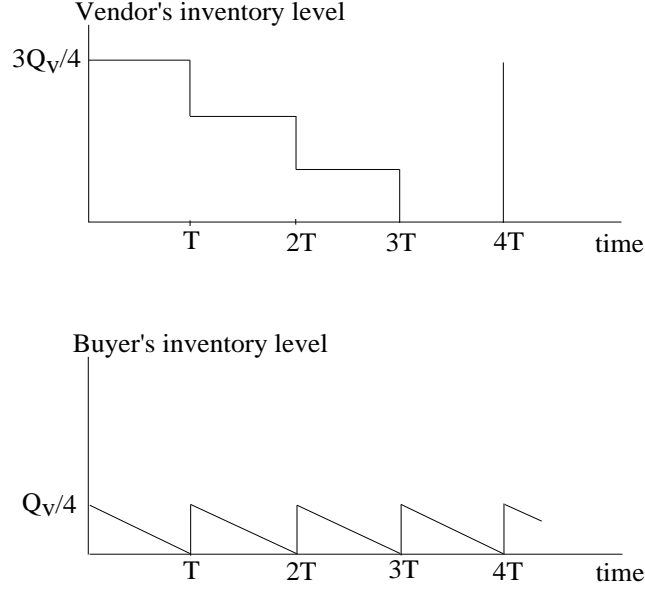
### III.1. Notation and Problem Formulation

Considering the case where the vendor replenishes from a supplier with ample supply and the demand rate at the buyer is a known constant, denoted by  $D$ , the problem is to compute the order quantities for the vendor and the buyer so that the total cost of the entire system is minimized. In this context, the vendor's order quantity, denoted by  $Q_v$ , represents the size of an inbound shipment to the vendor, whereas the buyer's order quantity, denoted by  $Q_b$ , represents the size of an outbound shipment from the

vendor. A summary of the notation used is given next.

- $K_v$ : Fixed replenishment cost of the vendor.
- $h_v$ : Holding cost per-unit per-unit-time for the vendor.
- $K_b$ : Fixed replenishment cost of the buyer.
- $h_b$ : Holding cost per-unit per-unit-time for the buyer.
- $h'$ : Echelon holding cost ( $h' = h_b - h_v > 0$ ).
- $R$ : Fixed cost per truck/cargo.
- $P$ : Truck/cargo capacity.
- $D$ : Buyer's demand rate.
- $Q_v$ : Vendor replenishment quantity, i.e., inbound shipment size.
- $Q_b$ : Buyer replenishment quantity, i.e., outbound shipment size.
- $T_v$ : Vendor replenishment cycle length.
- $T_b$ : Buyer replenishment cycle length.
- $n$ : Number of buyer replenishments within a vendor replenishment cycle ( $T_v = nT_b$ , and thus  $Q_b = Q_v/n$ ).

The replenishment costs, and hence the inbound/outbound transportation costs, exhibit economies of scale. The vendor replenishes itself and the buyer periodically so that transportation scale economies can be achieved. The time between two successive vendor replenishments represents the vendor replenishment cycle whereas the time between two successive buyer replenishments represents the buyer replenishment cycle. Hence, the buyer's replenishment cycle length, denoted by  $T_b$ , is given by  $Q_b/D$ . The vendor's replenishment cycle length, denoted by  $T_v$ , is given by  $T_v = nT_b$  where  $n$  is a positive integer denoting the number of buyer replenishments within a replenishment cycle of the vendor. Under these assumptions, inventory profiles of the vendor and buyer are illustrated in Figure 9.

Figure 9 Inventory Load Profiles of the Vendor and the Buyer ( $n = 4$ )

It is easy to show that the average annual total costs for Models I and II, denoted by  $G(n, Q_v)$  and  $\hat{G}(n, Q_v)$ , respectively, are given by

$$\text{Model I: } G(n, Q_v) = C_v(Q_v) \frac{D}{Q_v} + \frac{h_v(n-1)Q_v}{2n} + \frac{nK_b D}{Q_v} + \frac{h_b Q_v}{2n},$$

$$\text{Model II: } \hat{G}(n, Q_v) = C_v(Q_v) \frac{D}{Q_v} + \frac{h_v(n-1)Q_v}{2n} + nC_b(Q_v/n) \frac{D}{Q_v} + \frac{h_b Q_v}{2n},$$

where  $C_v(Q_v)$  and  $C_b(Q_b)$  are as expressed in (3.1) and (3.2), respectively. Hence, Model I incorporates the general replenishment cost function and the truck/cargo capacity constraints for inbound transportation only, whereas Model II incorporates these for inbound and outbound transportation, simultaneously. Noting that  $D/Q_v$  gives the number of replenishments per year, the first terms in the above cost expressions are the annual replenishment cost and the truck/cargo costs used for inbound transportation. The second terms of these expressions give the annual inventory

holding costs at the vendor. The third and fourth terms are the retailer's annual replenishment/truck and inventory holding costs, respectively.

Rewriting the average annual cost functions for Models I and II, we have

$$\begin{aligned} \text{Model I: } G(n, Q_v) &= \frac{K_v D}{Q_v} + \left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} + \frac{h_v Q_v}{2} + \frac{n K_b D}{Q_v} + \frac{h' Q_v}{2n}, \\ \text{Model II: } \hat{G}(n, Q_v) &= \frac{K_v D}{Q_v} + \left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} + \frac{h_v Q_v}{2} + \frac{n K_b D}{Q_v} + \left\lceil \frac{Q_v}{nP} \right\rceil \frac{nRD}{Q_v} + \frac{h' Q_v}{2n}. \end{aligned}$$

The problems under consideration are then given by:

	<b>Model I</b>		<b>Model II</b>
min	$G(n, Q_v)$	min	$\hat{G}(n, Q_v)$
s.t.	$Q_v \geq 0$	s.t.	$Q_v \geq 0$
	$n : \quad \text{a positive integer.}$		$n : \quad \text{a positive integer.}$

Next, we develop heuristic and exact optimization procedures for Models 1 and 2 in Sections III.2 and III.3, respectively.

### III.2. Model I: A Model with Explicit Inbound Costs

We begin by presenting a basic algorithm that can be used to solve the classical EOQ problem where the replenishment cost is given by Equation (3.1). This basic algorithm is based on a previous paper by Lee (1986) who provides a solution technique for the EOQ problem involving a replenishment cost function of the form  $C(Q)$ . As we discuss below, a special case of Lee's algorithm (Lee 1986) can be used as a building block for solving the problems of interest in this paper.

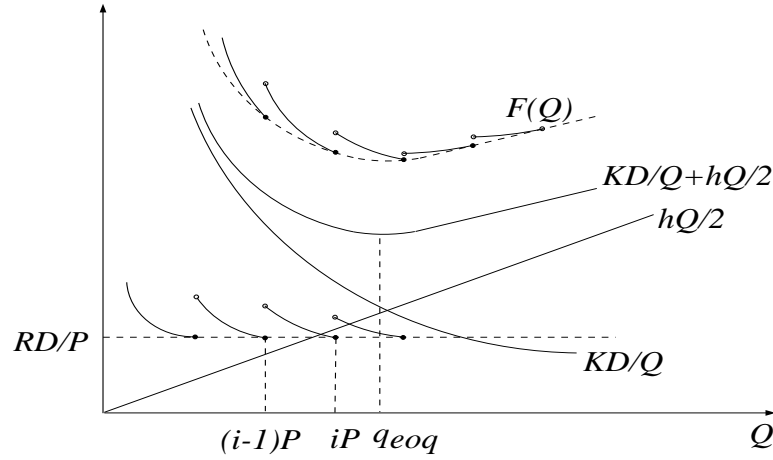
Now, consider the classical EOQ problem with an annual demand  $D$ , per-unit, per-unit-time holding cost  $h$ , and a general replenishment cost given by  $K + \lceil Q/P \rceil R$ ,

where  $Q$  is the order quantity. Hence, the average annual total cost, denoted  $F(Q)$ , is given by

$$F(Q) = \frac{KD}{Q} + \frac{hQ}{2} + \left\lceil \frac{Q}{P} \right\rceil \frac{RD}{Q}. \quad (3.3)$$

A graphical illustration of  $F(Q)$  is given in Figure 10. Obviously, the sum of the first two terms of  $F(Q)$ , denoted by  $H(Q)$ , is an EOQ-type convex function of  $Q$  with a minimizer at  $q_{eoq} = \sqrt{2KD/h}$ . The third term of (3.3) represents the annual cargo cost with a minimum value, equal to  $R(D/P)$ , at  $Q = kP$  for all positive integers  $k$ . Also, for each fixed  $k$ , this term is a decreasing convex function of  $Q$  over  $(k-1)P < Q \leq kP$ . Knowing these characteristics of  $F(Q)$ , it is straightforward to verify that the following properties are satisfied.

Figure 10 An Illustration of  $F(Q)$ .



**PROPERTY 1** Over  $(k-1)P < Q \leq kP$ ,  $k = 1, 2, \dots$ , function  $F(Q)$  in (3.3) reduces to

$$F(Q) = \frac{KD}{Q} + \frac{kRD}{Q} + \frac{hQ}{2},$$

and hence it is an EOQ-type function with a stationary point at

$$Q_k = \sqrt{2(K + kR)D/h}.$$

We say that  $Q_k$  is realizable if  $(k - 1)P < Q_k \leq kP$  or  $\lceil Q_k/P \rceil = k$ .

**PROPERTY 2** Let  $i$  denote the integer such that  $iP < q_{eoq} \leq (i + 1)P$ . For all  $k \leq i$ ,  $F(Q)$  in (3.3) is decreasing over  $(k - 1)P < Q \leq kP$ . Hence, if  $k \leq i$ , then  $Q_k$  is not realizable and  $F(Q) \geq F(kP)$  over  $Q \leq kP$ .

**PROPERTY 3** If  $k \geq i + 1$ , then  $F(Q) \geq F(kP)$  over  $Q \geq kP$ .

**PROPERTY 4** If  $Q_{i+1} \geq (i + 1)P$ , then  $F(Q)$  in (3.3) is decreasing over  $iP < Q \leq (i + 1)P$ . On the other hand, if  $Q_{i+1} < (i + 1)P$ , then  $F(Q)$  is decreasing over  $iP < Q \leq Q_{i+1}$  and increasing over  $Q_{i+1} \leq Q \leq (i + 1)P$ .

The following algorithm, a simplified version of Lee (1986), builds on Properties 1–4, and it can be used to find the optimal  $Q$  minimizing  $F(Q)$  in (3.3). A detailed proof of the optimality of this algorithm is provided in Lee (1986).

**ALGORITHM 1 – A Modification of Lee’s Algorithm in Lee (1986)**

Step 1. Compute  $q_{eoq} = \sqrt{2KD/h}$ .

Step 2. Let  $i$  denote the integer such that  $iP < q_{eoq} \leq (i + 1)P$ . Compute

$$Q_{i+1} = \sqrt{\frac{2[K + (i + 1)R]D}{h}}.$$

If  $Q_{i+1} \geq (i + 1)P$ , then go to Step 3. Otherwise go to Step 4.

Step 3. Compute the cost for  $Q = iP$  and  $Q = (i + 1)P$ .

Select the one that yields the minimum cost as the optimal  $Q$  and stop.



Step 4. Compute the cost for  $Q = iP$  and  $Q = Q_{i+1}$ .

Select the one that yields the minimum cost as the optimal  $Q$  and stop.

Observe that the optimal solution produced by this algorithm is either equal to  $Q_{i+1}$  or  $iP$  or  $(i+1)P$  where  $i$  is as defined in Step 2 above. That is, the minimizer of  $F(Q)$  in (3.3) is given by

$$\arg \min \{F(Q_{i+1}), F(iP), F((i+1)P)\}.$$

In fact, if we combine the result stated in the above expression with the results stated in Properties 1–4, we conclude that

$$\arg \min \{F(\min \{Q_{i+1}, (i+1)P\}), F(iP)\}. \quad (3.4)$$

For future reference, let us also revisit the classical EOQ problem where we minimize

$$H(Q) = \frac{KD}{Q} + \frac{hQ}{2},$$

and find that  $q_{eoq} = \sqrt{2KD/h}$  is the optimal solution. It is easy to show that

$$\frac{H(Q)}{H(q_{eoq})} = \frac{H(Q)}{\sqrt{2KDh}} = \frac{1}{2} \left( \frac{Q}{q_{eoq}} + \frac{q_{eoq}}{Q} \right). \quad (3.5)$$

Now, suppose that  $q_{eoq} \geq P$ . Then either one of the following two cases is true:

**PROPERTY 5**

- **Part i)**  $iP \leq q_{eoq} \leq \sqrt{i(i+1)}P$ . In this case, using an EOQ value of  $iP$  implies the following error:

$$\frac{H(iP)}{H(q_{eoq})} = \frac{1}{2} \left( \frac{q_{eoq}}{iP} + \frac{iP}{q_{eoq}} \right).$$

Since the above function is increasing in  $q_{eoq}$  in this range, its highest value is

attained when  $Q = \sqrt{i(i+1)}P$ . Therefore,

$$\frac{H(iP)}{H(q_{eoq})} = \frac{1}{2} \left( \frac{q_{eoq}}{iP} + \frac{iP}{q_{eoq}} \right) \leq \frac{1}{2} \left( \sqrt{\frac{i}{i+1}} + \sqrt{\frac{i+1}{i}} \right) \leq \frac{1}{2} \left( \frac{1}{\sqrt{2}} + \sqrt{2} \right) \leq 1.06. \quad (3.6)$$

- **Part ii)**  $\sqrt{i(i+1)}P \leq q_{eoq} \leq (i+1)P$ . In this case, using an EOQ value of  $(i+1)P$  implies the following error:

$$\frac{H((i+1)P)}{H(q_{eoq})} = \frac{1}{2} \left( \frac{q_{eoq}}{(i+1)P} + \frac{(i+1)P}{q_{eoq}} \right).$$

Since the above function is decreasing in  $q_{eoq}$  in the range considered, its highest value is attained when  $q_{eoq} = \sqrt{i(i+1)}P$ . Therefore,

$$\frac{H((i+1)P)}{H(q_{eoq})} = \frac{1}{2} \left( \frac{q_{eoq}}{(i+1)P} + \frac{(i+1)P}{q_{eoq}} \right) \leq \frac{1}{2} \left( \sqrt{\frac{i}{i+1}} + \sqrt{\frac{i+1}{i}} \right) \leq 1.06.$$

Having developed the fundamental Properties 2 through 5 that will be used in the remaining portion of this chapter, let us revisit the objective function of Model I. We define

$$A_n = K_v + nK_b, \quad B_n = h_v + h'/n, \quad n = 1, 2, \dots, \quad (3.7)$$

$$\text{so that} \quad G(n, Q_v) = \frac{A_n D}{Q_v} + \frac{B_n Q_v}{2} + \left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v}. \quad (3.8)$$

Observe that, for a fixed  $n$ , the above function has the same characteristics as the function  $F(\cdot)$  defined earlier. Hence, given  $n$ , the minimizer of (3.8) can be computed using Algorithm 1 by letting  $K = A_n$ ,  $h = B_n$ , and  $Q = Q_v$ . As a result, if we can find an upper bound, denoted  $n_{max}$ , on the optimal value of  $n$ , then we can find the optimal solution of Model I in a finite number of steps by computing the minimizers of  $G(n, Q_v)$  for  $n = 1, 2, \dots, n_{max}$  using Algorithm 1. Hence, in the following section, we first develop a heuristic solution procedure for Model I. Later, we use the cost of the heuristic solution and develop a technique for computing  $n_{max}$ . As we prove in

the following subsection, the error bound of the heuristic method is 6%. Thus, it can effectively replace the exact optimal solution.

### III.2.1. Heuristic Approach for Model I

Recalling the objective function  $G(n, Q_v)$  of Model I, we define

$$F_I(Q_v) = \frac{K_v D}{Q_v} + \left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} + \frac{h_v Q_v}{2}, \quad \text{and} \quad H_I(Q_b) = \frac{K_b D}{Q_b} + \frac{h' Q_b}{2},$$

where  $Q_b = Q_v/n$ . For the sake of convenience, let us treat  $n$  as a continuous variable for the moment. It follows that

$$G(n, Q_v) = F_I(Q_v) + H_I(Q_v/n)$$

which is equivalent to our original cost function. Now, observe that function  $F_I(Q_v)$  can be minimized over  $Q_v \geq 0$  using Algorithm 1 by letting  $K = K_v, h = h_v$ , and  $Q = Q_v$ . Let  $Q_I$  denote the resulting minimizer of  $F_I(Q_v)$ . Recalling Expression (3.4), we can write

$$Q_I = \arg \min \{F_I(\min \{Q_{i+1}, (i+1)P\}), F_I(iP)\}. \quad (3.9)$$

where  $i$  is the integer satisfying

$$iP < \sqrt{\frac{2K_v D}{h_v}} \leq (i+1)P.$$

Also, note that  $H_I(Q_b)$  is an EOQ-type function. Its minimizer, denoted  $q_I$ , is given by

$$q_I = \sqrt{\frac{2K_b D}{h'}}, \quad \text{so that} \quad H_I(Q_I) = \sqrt{2K_b h' D}.$$

It follows that

$$H_I(Q_b) \geq \sqrt{2K_b h' D}, \quad \forall Q_b \geq 0.$$

As a consequence,  $F_I(Q_I) + \sqrt{2K_b h' D}$  is a lower bound on the cost function of our problem, i.e.,

$$\min_{n, Q_v} G(n, Q_v) \geq F_I(Q_I) + \sqrt{2K_b h' D}. \quad (3.10)$$

Now, we are ready to state our heuristic algorithm.

**ALGORITHM 2 – A Heuristic Algorithm for Model I**

Find the  $Q_I$  that minimizes  $F_I(Q_v)$  using Algorithm 1 where  $K = K_v$  and  $h = h_v$ .

**Case 1:** If  $Q_I > q_I$ , then let  $m$  denote the integer that satisfies

$$\sqrt{m(m-1)} < \frac{Q_I}{q_I} \leq \sqrt{m(m+1)}.$$

Compute  $G(m, Q_v)$  by substituting  $n = m$  in Expressions (3.7) and (3.8). Minimize  $G(m, Q_v)$  using Algorithm 1 where  $K = A_m = K_v + mK_b$  and  $h = B_m = h_v + h'/m$ , and obtain the heuristic solution for  $Q_v$ .

**Case 2:** If  $Q_I \leq q_I$ , then set  $n = 1$  in Expressions (3.7) and (3.8). Minimize the resulting  $G(1, Q_v)$  using Algorithm 1 where  $K = A_1 = K_v + K_b$  and  $h = B_1 = h_v + h'$ , and find the heuristic solution for  $Q_v$ .

**THEOREM 1**  $\bar{G}_I/G^* < 1.06$  where  $\bar{G}_I$  is the cost of the heuristic solution obtained by using Algorithm 2 and  $G^*$  is the optimal cost for Model I.

**Proof:**

**Case 1:**  $Q_I > q_I$ .

The suggested heuristic solution is given by  $\min_{Q_v} G(m, Q_v)$ , and, as a consequence, we have  $\min_{Q_v} G(m, Q_v) \leq G(m, Q_I)$ . In order to complete the proof, we will show that

$$G(m, Q_I) \leq 1.06 \left( F_I(Q_I) + \sqrt{2K_b h' D} \right).$$

However, since  $G(m, Q_I) = F_I(Q_I) + H_I(Q_I/m)$ , it is sufficient to show  $H_I(Q_I/m) \leq 1.06\sqrt{2K_b h' D}$ . Recalling Property 5, observe that

$$\frac{H_I(Q_I/m)}{\sqrt{2K_b D h'}} = \frac{\frac{K_b m D}{Q_I} + \frac{h' Q_I}{2m}}{\sqrt{2K_b D h'}} = \frac{m}{Q_I} \sqrt{\frac{K_b D}{2h'}} + \frac{Q_I}{2m} \sqrt{\frac{h'}{2K_b D}} = \frac{1}{2} \left( \frac{m}{Q_I/q_I} + \frac{Q_I/q_I}{m} \right).$$

By definition  $\sqrt{m(m-1)} < Q_I/q_I \leq \sqrt{m(m+1)}$ , and thus there may be two possibilities:

**Case 1.1:**  $\sqrt{m(m-1)} < Q_I/q_I \leq m$ .

In this case, the highest value that  $H_I(Q_I/m)/\sqrt{2K_b D h'}$  can attain is at  $Q_I/q_I = \sqrt{m(m-1)}$ . However, since  $Q_I > q_I$ , we have

$$\begin{aligned} \frac{H_I(Q_I/m)}{\sqrt{2K_b D h'}} = \frac{1}{2} \left( \frac{m}{Q_I/q_I} + \frac{Q_I/q_I}{m} \right) &\leq \frac{1}{2} \left( \sqrt{\frac{m-1}{m}} + \sqrt{\frac{m}{m-1}} \right) \\ &\leq \frac{1}{2} \left( \sqrt{2} + \frac{1}{\sqrt{2}} \right) \approx 1.06. \end{aligned} \quad (3.11)$$

**Case 1.2:**  $m < Q_I/q_I \leq \sqrt{m(m+1)}$ .

In this case, the highest value that  $H_I(Q_I/m)/\sqrt{2K_b D h'}$  can attain is at  $Q_I/q_I = \sqrt{m(m+1)}$ . Hence,

$$\begin{aligned} \frac{H_I(Q_I/m)}{\sqrt{2K_b D h'}} = \frac{1}{2} \left( \frac{m}{Q_I/q_I} + \frac{Q_I/q_I}{m} \right) &\leq \frac{1}{2} \left( \sqrt{\frac{m+1}{m}} + \sqrt{\frac{m}{m+1}} \right) \\ &\leq \frac{1}{2} \left( \sqrt{2} + \frac{1}{\sqrt{2}} \right) \approx 1.06. \end{aligned} \quad (3.12)$$

Our analysis for Case 1.1 and Case 1.2 leads to

$$G(m, Q_I) \leq 1.06 \left( F_I(Q_I) + \sqrt{2K_b h' D} \right)$$

so that

$$\bar{G}_I = \min_{Q_v} G(m, Q_v) \leq G(m, Q_I) \leq 1.06G^*.$$

**Case 2:**  $Q_I \leq q_I$ .

Note that for a given value of  $Q_v$ , function  $G(n, Q_v)$  is convex in  $n$ . Also, the first order conditions for optimality dictate that  $\partial G(n, Q_v)/\partial n = 0$ , which requires any solution  $(n, Q_v)$  to satisfy

$$n(Q_v) = \frac{Q_v}{\sqrt{\frac{2K_b D}{h'}}} = \frac{Q_v}{q_I}. \quad (3.13)$$

Considering the integer requirements on  $n$  and noting that the proposed heuristic value of  $n$  is 1, we analyze the objective function over two regions:

First, we consider those  $Q_v$  such that  $Q_v \leq q_I$ . Over this region, the minimum of our objective function is achieved at  $n = 1$ . This is because  $H_I(\cdot)$  is convex with a unique minimizer at  $q_I$  so that

$$G(1, Q_v) = F_I(Q_v) + H_I(Q_v) \leq G(n, Q_v) = F_I(Q_v) + H_I(Q_v/n), \quad \forall n > 1, Q_v \leq q_I.$$

Therefore,

$$G_I(1, Q_v) \leq G(n, Q_v), \quad \forall n \text{ and } \forall Q_v \leq q_I. \quad (3.14)$$

Next, we consider those  $Q_v$  such that  $Q_v > q_I$ , and we define  $i$  as the integer satisfying  $iP < q_I \leq (i+1)P$ . Again, we have two possibilities. Namely,  $i \geq 1$  or  $i = 0$ :

**Case 2.1:**  $Q_v > q_I$ ,  $iP < q_I \leq (i+1)P$ ,  $i \geq 1$ .

In this case, either one of the following is true:

- **Case 2.1.1:**  $iP < q_I \leq \sqrt{i(i+1)}P$  so that  $H_I(iP) \leq H_I((i+1)P)$ .

Similar to Equation (3.6), we can show that

$$H_I(iP) \leq 1.06H_I(q_I) = 1.06\sqrt{2K_b h' D}.$$

First, suppose we also have  $Q_I \leq iP$ . Then we have  $q_I > iP \geq Q_I$ , and thus  $F_I(Q_v) > F_I(iP)$ , for all  $Q_v \geq iP$ . Hence,

$$G(1, iP) \leq F_I(Q_I) + 1.06\sqrt{2K_b h' D}, \quad \forall Q_v > q_I. \quad (3.15)$$

Now, suppose  $Q_I > iP$  so that we have  $iP < Q_I \leq q_I \leq \sqrt{i(i+1)}P$  where  $i \geq 1$ . For this particular case, we have

$$Q_I = Q_{i+1} = \sqrt{\frac{(K_v + (i+1)R)D}{h_v}}, \quad \text{and hence, } F_I(Q_I) = F_I(Q_{i+1}).$$

Defining

$$F_I^{i+1}(Q_v) = K_v \frac{D}{Q_v} + (i+1) \frac{RD}{Q_v} + \frac{h_v}{2} Q_v.$$

we can write  $F_I^{i+1}(iP) \leq 1.06F_I(Q_{i+1})$ . Also, note that

$$F_I(iP) < F_I^{i+1}(iP) \leq 1.06F_I(Q_{i+1}) = 1.06F_I(Q_I).$$

Thus, if  $Q_I > iP$  then

$$G(1, iP) \leq 1.06 \left( F_I(Q_I) + \sqrt{2K_b h' D} \right). \quad (3.16)$$

- **Case 2.1.2:**  $\sqrt{i(i+1)}P < q_I \leq (i+1)P$  so that  $H_I(iP) > H_I((i+1)P)$ .

In this case, recalling the second part of Property 5, we can write

$$H_I((i+1)P) \leq 1.06\sqrt{2K_b h' D}.$$

Since  $Q_I < q_I$ , we also have  $Q_I < (i+1)P$ . Let us consider those  $Q_v$  such that  $Q_v \leq (i+1)P$ , and hence  $\sqrt{i(i+1)}P < q_I < Q_v \leq (i+1)P$ . Under these conditions, we have

$$H_I(Q_v) \leq H_I(iP) \leq H_I\left(\frac{Q_v}{n}\right),$$

so that  $\forall q_I < Q_v \leq (i+1)P$ , we can write

$$G(1, Q_v) = F_I(Q_v) + H_I(Q_v) \leq F_I(Q_v) + H_I\left(\frac{Q_v}{n}\right) = G(n, Q_v), \quad (3.17)$$

Next, let us consider those  $Q_v$  such that  $Q_v \geq (i+1)P$ . In this case, we have  $Q_I < q_I \leq (i+1)P \leq Q_v$ . It follows by Property 3 that  $F_I(Q_v) \geq F_I((i+1)P)$ , for all  $Q_v \geq (i+1)P$ , and, therefore,

$$G(1, (i+1)P) \leq F_I(Q_v) + 1.06\sqrt{2K_b h' D}, \quad \forall Q_v \geq (i+1)P. \quad (3.18)$$

**Case 2.2:**  $Q_v > q_I$ ,  $iP < q_I \leq (i+1)P$ ,  $i = 0$ .

In this case, we have  $Q_I < q_I \leq P$ . Let us first consider those values of  $Q_v$  such that  $Q_v \leq P$ , and thus  $Q_I < q_I \leq Q_v \leq P$ . Observe that, for all  $Q_v$  within this region  $F_I(q_I) < F_I(Q_v)$ . It follows that under the assumptions of this case, if we also have  $Q_v \leq P$

$$G(1, q_I) = F_I(q_I) + H_I(q_I) = F_I(q_I) + \sqrt{2K_b h' D} \leq F_I(Q_v) + \sqrt{2K_b h' D}, \quad (3.19)$$

$\forall Q_v$  s.t.  $q_I \leq Q_v \leq P$ .

Finally, consider those values of  $Q_v$  such that  $Q_v > P$  so that we can analyze the case where  $Q_I < q_I \leq P < Q_v$ . For this specific case, by Properties 3 and 4, we have  $F_I(P) \leq F_I(Q_v)$  and  $F_I(q_I) \leq F_I(P)$ . Consequently,

$$G(1, q_I) = F_I(q_I) + H_I(q_I) \leq F_I(Q_v) + \sqrt{2K_b h' D}, \quad \forall Q_v > P. \quad (3.20)$$

Combining our results given in (3.14)–(3.20), we conclude that  $\min_{Q_v} G(1, Q_v)$  is less than, or equal to, the cost of any feasible solution under the assumptions of Case 2. ■

**COROLLARY 1**  $F_I(Q_I) + \sqrt{2K_b h' D} \leq G^* \leq \bar{G}_I \leq 1.06 (F_I(Q_I) + \sqrt{2K_b h' D})$ .



**Proof:** The corollary follows from Theorem 1 and Expression (3.10). ■

**REMARK 1** Theorem 1 states that the coordinated replenishment policy obtained using Algorithm 2 has a cost that is at most 1.06 times the lower bound. This, in turn, implies that the suggested heuristic is very efficient for all practical purposes. There are other efficient heuristic procedures, with similarly promising error bounds, in the existing literature for inventory coordination problems with deterministic constant demand but without cargo capacities and costs. These include the heuristic solutions for the one-warehouse multi-retailer problem (Bramel and Simchi-Levi 1997, pp. 158–162) and the joint replenishment problem (Zipkin 2000, pp. 149–152). It is worthwhile to note that the error bound analyses of these existing heuristics resemble the error bound analysis of Algorithm 2. This is simply because, for problems without or with cargo capacities and costs, both approaches rely on the neat analytical properties, such as those given in Property 5, of EOQ-type cost functions. Although the corresponding total cost function,  $G(n, Q_v)$ , for the problem with cargo capacities and costs is discontinuous, each piece of this function is of the EOQ-type. Hence, the error bound analysis presented in the proof of Theorem 1 relies heavily on Property 5.

◆

### III.2.2. Exact Solution for Model I

In this section, we provide an upper bound, denoted  $n_{max}$ , on the optimal value of  $n$ , and develop a finite time exact algorithm for Model I. To this end, we utilize Corollary 1 which provides an upper bound on the optimal cost of our problem.

Let us define  $g(n, Q_v)$  as follows:

$$g(n, Q_v) = \frac{(K_v + nK_b)D}{Q_v} + \left(h_v + \frac{h'}{n}\right) \frac{Q_v}{2} + \frac{RD}{P}.$$

Recalling the objective function of Model I, observe that

$$G(n, Q_v) = \frac{(K_v + nK_b)D}{Q_v} + \frac{h_v Q_v}{2} + \left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} + \frac{h' Q_v}{2n} \geq g(n, Q_v), \quad \forall n, Q_v. \quad (3.21)$$

That is, function  $g(n, Q_v)$  is, in fact, a lower bound on the objective function of our problem. Let  $g^* = \min_{n, Q_v} g(n, Q_v)$ , i.e.,  $g^*$  denotes the minimum value that  $g(n, Q_v)$  can take. It follows from Corollary 1 and Expression (3.21) that

$$g^* \leq G^* \leq 1.06 \left( F_I(Q_I) + \sqrt{2K_b h' D} \right). \quad (3.22)$$

Observe that, for a given  $n$ , function  $g(n, Q_v)$  is convex in  $Q_v$  with a minimizer at  $q^*(n) = \sqrt{2(K_v + nK_b)D/(h_v + h'/n)}$ . Thus, for any given  $n$ ,

$$g(n, q^*(n)) = \sqrt{2(K_v + nK_b)(h_v + h'/n)D} + \frac{DR}{P} \leq g(n, Q_v), \quad \forall Q_v. \quad (3.23)$$

Also, it can be easily shown that  $g(n, q^*(n))$  is decreasing over  $n \leq \lfloor n_0 \rfloor$  and increasing over  $n > \lfloor n_0 \rfloor$  where  $n_0 = \sqrt{K_v h' / K_b h_v}$  so that Lemma 1 follows.

**LEMMA 1** The optimal  $n$  for Model I lies over  $[n_{min}, n_{max}]$  where

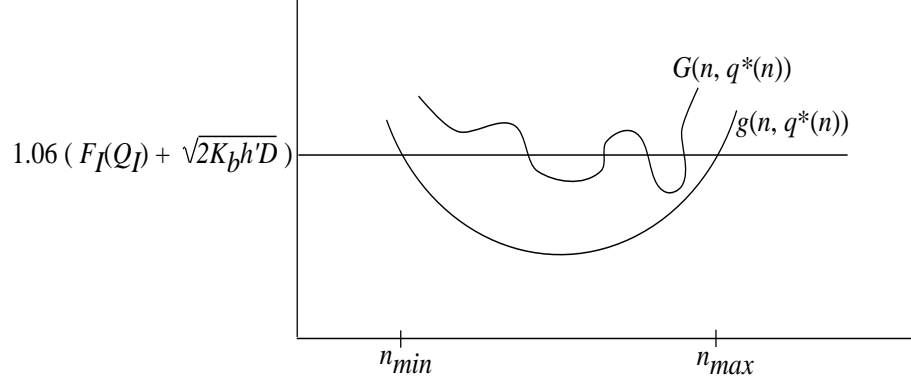
$$\begin{aligned} n_{min} &= \max \left\{ \left\lceil \frac{N - \sqrt{N^2 - 4K_v K_b h_v h'}}{2K_b h_v} \right\rceil, 1 \right\}, \\ n_{max} &= \left\lceil \frac{N + \sqrt{N^2 - 4K_v K_b h_v h'}}{2K_b h_v} \right\rceil, \quad \text{and} \\ N &= \frac{[1.06 (F_I(Q_I) + \sqrt{2K_b h' D}) - RD/P]^2}{2D} - K_v h_v - K_b h'. \end{aligned}$$

**Proof:** Combining Expressions (3.22) and (3.23), we conclude that the optimal  $n$  satisfies

$$g(n, q^*(n)) \leq 1.06 \left( F_I(Q_I) + \sqrt{2K_b h' D} \right). \quad (3.24)$$

A graphical illustration of this argument is given in Figure 11.

Figure 11 An Illustration of the Idea Behind Algorithm 3



Setting the expression of  $g(n, q^*(n))$  equal to  $1.06(F_I(Q_I) + \sqrt{2K_b h' D})$  and solving for  $n$ , we obtain the expressions of  $n_{min}$  and  $n_{max}$  given in Lemma 1. Also, since  $N^2 \geq 4K_v K_b h_v h'$  it can be easily verified that  $n_{min} \leq n_0 \leq n_{max}$ . Recalling that  $g(n, q^*(n))$  is decreasing over  $n \leq \lfloor n_0 \rfloor$  and increasing over  $n > \lceil n_0 \rceil$ , we conclude that Expression (3.24) holds over  $[n_{min}, n_{max}]$ . ■

### ALGORITHM 3 – Optimal Algorithm for Model I

For  $n = n_{min}, n_{min} + 1, \dots, n_{max}$ , compute  $G(n, Q_v)$  using Expression (3.8) where  $A_n$  and  $B_n$  are given by (3.7). For each  $G(n, Q_v)$ , execute Algorithm 1 and obtain the value of  $Q_v$  minimizing  $G(n, Q_v)$ . The optimal solution for Model I is the  $(n, Q_v)$  pair corresponding to  $\min \{ \min_{Q_v \geq 0} G(n, Q_v) : n = n_{min}, \dots, n_{max} \}$ .

**COROLLARY 2** Both  $n_{min}$  and  $n_{max}$  are finite positive integers, and it follows that Algorithm 3 is a finite time exact algorithm for Model I.

### III.3. Model II: A Model with Explicit Inbound and Outbound Costs

Building on our results in Section III.2, we now analyze Model II. Let us rewrite the objective function of Model II as follows:

$$\hat{G}(n, Q_v) = \frac{(K_v + nK_b)D}{Q_v} + \left(h_v + \frac{h'}{n}\right) \frac{Q_v}{2} + \left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} + \left\lceil \frac{Q_v}{nP} \right\rceil \frac{nRD}{Q_v}. \quad (3.25)$$

Let  $\hat{G}_n(Q_v)$  denote function (3.25) for a given value of  $n = 1, 2, \dots$ . The following algorithm can be used to optimize  $\hat{G}_n(Q_v)$  over  $Q_v \geq 0$ .

#### ALGORITHM 4 – An Extension of Algorithm 1

Step 1. Given  $n$ , compute  $Q_0 = \sqrt{\frac{2(K_v + nK_b)D}{h_v + h'/n}}$ .

Step 2. Let  $i$  denote the integer satisfying  $iP < Q_0 \leq (i+1)P$ , and let  $j = \left\lceil \frac{i+1}{n} \right\rceil$ .

For  $k = i + 1, \dots, nj$ , compute

$$Q_k = \sqrt{\frac{2[K_v + nK_b + (k + nj)R]D}{h_v + h'/n}}.$$

Step 3. Compute  $\hat{G}_n(Q_v)$ , for  $Q_v = n(j-1)P, iP, (i+1)P, \dots, njP$ , and for those  $Q_k, k = i+1, \dots, nj$ , such that  $(k-1)P < Q_k \leq kP$ , i.e., for realizable  $Q_k$  values. Among these alternative values of  $Q_v$ , select the one that yields the minimum cost as the minimizer (Note that if  $j-1 = 0$ , then there is no need to calculate the cost for  $Q_v = n(j-1)P$ . Similarly, if  $i = 0$ , then there is no need to calculate the cost for  $Q_v = iP$ ).

**THEOREM 2** For a given  $n$ , Algorithm 4 finds the minimizer of  $\hat{G}_n(Q_v)$ .

**Proof:** The proof is based on the following properties (i.e. Properties 6, 7 and 8) of  $\hat{G}_n(Q_v)$ .

In order to show that Properties 6–8 are true, let us recall Expression (3.25) and define

$$\hat{H}(Q_v) = \frac{(K_v + nK_b)D}{Q_v} + \left(h_v + \frac{h'}{n}\right) \frac{Q_v}{2},$$

so that

$$\hat{G}_n(Q_v) = \hat{H}(Q_v) + \left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} + \left\lceil \frac{Q_v}{nP} \right\rceil \frac{nRD}{Q_v}. \quad (3.26)$$

Observe that  $\hat{H}(Q_v)$  is an EOQ-type convex function with a unique minimizer at  $Q_0$  where  $Q_0$  is as defined in Step 1. Also, observe that the second term of (3.26) is minimized at  $Q_v = kP$  for some positive integer  $k$  and its minimum value is  $RD/P$ . Similarly, the third term of (3.26) is minimized at  $Q_v = knP$  for some positive integer  $k$  and its minimum value is again  $RD/P$ . It follows that

$$\left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} + \left\lceil \frac{Q_v}{nP} \right\rceil \frac{nRD}{Q_v} \geq \frac{2RD}{P}, \quad \forall Q_v \geq 0. \quad (3.27)$$

**PROPERTY 6**  $\hat{G}_n(Q_v) \geq \hat{G}_n(n(j-1)P)$ ,  $\forall Q_v \leq n(j-1)P$ .

**Proof:** Now, consider those  $Q_v$  such that  $Q_v \leq n(j-1)P$ . Recall that by definition  $n(j-1)P$  is the least multiple of  $nP$  less than  $Q_0$ . Hence,  $Q_v \leq n(j-1)P < Q_0$ . This in turn implies that if  $Q_v \leq n(j-1)P$ , then  $\hat{H}(n(j-1)P) \leq \hat{H}(Q_v)$ . Since  $n(j-1)P$  is an integer multiple of both  $P$  and  $nP$ ,

$$\left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} + \left\lceil \frac{Q_v}{nP} \right\rceil \frac{nRD}{Q_v}$$

is minimized at  $Q_v = n(j-1)P$ . Therefore, if  $Q_v \leq n(j-1)P$  then

$$\hat{G}_n(Q_v) = \hat{H}(Q_v) + \left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} + \left\lceil \frac{Q_v}{nP} \right\rceil \frac{nRD}{Q_v} \geq \hat{G}_n(n(j-1)P) = \hat{H}(n(j-1)P) + \frac{2RD}{P}.$$

It follows that Property 6 is true, and thus the global minimizer of  $\hat{G}_n(Q_v)$  cannot be over  $Q_v < n(j-1)P$ . ■

**PROPERTY 7**  $\hat{G}_n(Q_v) \geq \hat{G}_n(njP)$ ,  $\forall Q_v \geq njP$ .

**Proof:** Next, we consider those  $Q_v$  such that  $Q_v \geq njP$ . By definition  $njP$  is the greatest multiple of  $nP$  greater than or equal to  $Q_0$ . Therefore,  $Q_v \geq njP$  implies that  $Q_v \geq njP \geq Q_0$ , and by the convexity of  $\hat{H}(Q_v)$ , we can write  $\hat{H}(Q_v) \geq \hat{H}(njP)$  for all  $Q_v \geq njP \geq Q_0$ . Using this inequality and (3.27) leads to

$$\hat{H}(Q_v) + \left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} + \left\lceil \frac{Q_v}{nP} \right\rceil \frac{nRD}{Q_v} \geq \hat{H}(njP) + \frac{2RD}{P}, \quad \forall Q_v \geq njP \geq Q_0.$$

It follows that  $\hat{G}_n(Q_v) \geq \hat{G}_n(njP)$  for all  $Q_v \geq njP$ , and thus there is no need to search for a global minimizer of  $\hat{G}_n(Q_v)$  over  $Q_v > njP$ . ■

**PROPERTY 8**  $\hat{G}_n(Q_v) \geq \hat{G}_n(iP), \quad \forall n(j-1)P < Q_v \leq iP.$

**Proof:** Let us consider those  $Q_v$  such that  $n(j-1)P < Q_v \leq iP$ . Since, by definition,  $iP < Q_0$ , we have  $n(j-1)P < Q_v \leq iP < Q_0$ . Using the fact that  $\hat{H}(Q_v)$  is convex and  $Q_0$  is its minimizer, we also have

$$\hat{H}(Q_v) \geq \hat{H}(iP), \quad \forall Q_v \text{ s.t. } n(j-1)P < Q_v \leq iP < Q_0. \quad (3.28)$$

Considering the third term in (3.26), we can show that

$$\left\lceil \frac{Q_v}{nP} \right\rceil \frac{nRD}{Q_v} = \frac{njRD}{Q_v}, \quad \text{if } n(j-1)P < Q_v < iP.$$

For  $Q_v = iP$ , since  $n(j-1)P < Q_v \leq iP < njP$ , this term is given by

$$\left\lceil \frac{Q_v}{nP} \right\rceil \frac{nRD}{Q_v} = \frac{njRD}{iP}.$$

We also have

$$\frac{njRD}{Q_v} \leq \frac{njRD}{iP}, \quad \forall Q_v \text{ s.t. } n(j-1)P < Q_v \leq iP. \quad (3.29)$$

Now, considering the second term in (3.26), we can show that this term realizes its minimum at  $Q_v = iP$ . Hence,

$$\left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} \geq \frac{RD}{P}, \quad \forall Q_v \text{ s.t. } n(j-1)P < Q_v \leq iP. \quad (3.30)$$

Combining Expressions (3.28), (3.29), and (3.30), we have

$$\hat{H}(Q_v) + \frac{njRD}{Q_v} + \left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} \geq \hat{H}(iP) + \frac{njRD}{iP} + \frac{RD}{P},$$

$$\forall Q_v \text{ s.t. } n(j-1)P < Q_v \leq iP.$$

Thus,  $G_n(Q_v) \geq G_n(iP)$ , for all  $Q_v$  such that  $n(j-1)P < Q_v \leq iP$ . ■

Hence, for computing its minimizer, we only need to evaluate  $\hat{G}_n(Q_v)$  over  $iP \leq Q_v \leq njP$  and at  $Q_v = n(j-1)P$ . Observe that the following property also holds.

**PROPERTY 9** Over  $(k-1)P < Q_v \leq kP$ ,  $k = i+1, \dots, nj$ , function  $\hat{G}_n(Q_v)$  is given by

$$\hat{G}_n(Q_v) = \frac{[K_v + nK_b + (k + nj)R]D}{Q_v} + \left( h_v + \frac{h'}{n} \right) \frac{Q_v}{2}.$$

It follows that  $\hat{G}_n(Q_v)$  is an EOQ-type function with a stationary point at  $Q_k$  where  $Q_k$  is defined as in Step 2 of Algorithm 4.

Consequently, the minimizer of  $\hat{G}_n(Q_v)$  is either at one of the realizable  $Q_k$  values or one of the breakpoints over  $[n(j-1)P, njP]$ . This completes the proof. ■

### III.3.1. Heuristic Approach for Model II

In this section we develop a heuristic approach for solving Model II. For this purpose, let us recall the objective function  $\hat{G}(n, Q_v)$  of Model II, given by Expression (3.25), and define

$$F_{II}(Q_v) = \left( K_v + \left\lceil \frac{Q_v}{P} \right\rceil R \right) \frac{D}{Q_v} + \frac{h_v Q_v}{2},$$

and

$$h(n, Q_v) = \left( K_b + \left\lceil \frac{Q_v}{nP} \right\rceil R \right) \frac{nD}{Q_v} + \frac{h' Q_v}{2n}.$$

It follows that

$$\hat{G}(n, Q_v) = F_{II}(Q_v) + h(n, Q_v)$$

which is equivalent to our original cost function. Now, observe that function  $F_{II}(Q_v)$  can be minimized over  $Q_v \geq 0$  using Algorithm 1. Let  $Q_{II}$  denote the resulting minimizer of  $F_{II}(Q_v)$ . We also define

$$H_{II}(Q_b) = \left( K_b + \left\lceil \frac{Q_b}{P} \right\rceil R \right) \frac{D}{Q_b} + \frac{h' Q_b}{2}, \quad (3.31)$$

and let  $q_{II}$  denote the minimizer of  $H_{II}(Q_b)$  over  $Q_b \geq 0$ . This minimizer can also be computed using Algorithm 1. Observe that

$$\min_{Q_b} H_{II}(Q_b) = H_{II}(q_{II}) \leq \min_{n, Q_v} h(n, Q_v).$$

As a consequence,  $F_{II}(Q_{II}) + H_{II}(q_{II})$  is a lower bound on the cost function of our problem, i.e.,

$$\min_{n, Q_v} G(n, Q_v) \geq F_{II}(Q_{II}) + H_{II}(q_{II}). \quad (3.32)$$

Now, we are ready to state our heuristic algorithm.

#### ALGORITHM 5 – A Heuristic Algorithm for Model II

Compute  $Q_{II}$  and  $q_{II}$  using Algorithm 1.

- **Case 1:** If  $q_{II} < Q_{II}$  and  $q_{II} < P$ , then let  $m = \left\lceil \frac{Q_{II}}{q_{II}} \right\rceil$ .

Compute  $\min_{Q_v} \hat{G}_m(Q_v)$  using Algorithm 4.

The heuristic solution is given by  $n = m$  and the resulting  $Q_v$  value.

- **Case 2:** If  $q_{II} < Q_{II}$  and  $q_{II} \geq P$ , then let  $m = \lfloor \frac{Q_{II}}{iP} \rfloor$  where  $i$  denotes the integer satisfying  $\sqrt{i(i-1)}P < q_{II} \leq \sqrt{i(i+1)}P$ .



Compute  $\min_{Q_v} \hat{G}_m(Q_v)$  using Algorithm 4.

The heuristic solution is given by  $n = m$  and the resulting  $Q_v$  value.

- **Case 3:** If  $Q_{II} \leq q_{II}$ , then set  $n = 1$  in Expression (3.25).

Compute  $\min_{Q_v} \hat{G}(1, Q_v)$  using Algorithm 1.

In this case, the heuristic solution is given by  $n = 1$  and the resulting  $Q_v$  value.

**THEOREM 3**  $\bar{G}_{II}/\hat{G}^* < 1.25$  where  $\bar{G}_{II}$  is the cost of the heuristic solution obtained by using Algorithm 5, and  $\hat{G}^*$  is the optimal cost for Model II.

**Proof:**

**Case 1:**  $q_{II} < Q_{II}$  and  $q_{II} < P$ .

In this case, the total cost of the heuristic solution is not greater than the cost of the solution given by  $(m, Q_{II})$  where

$$m = \left\lceil \frac{Q_{II}}{q_{II}} \right\rceil \quad \text{so that} \quad m \geq \frac{Q_{II}}{q_{II}}, \quad \text{and thus} \quad \frac{Q_{II}}{m} \leq q_{II}. \quad (3.33)$$

Under the assumptions of this case, we also have

$$2 \leq m = \left\lceil \frac{Q_{II}}{q_{II}} \right\rceil < \frac{Q_{II}}{q_{II}} + 1 \quad \text{so that} \quad q_{II} < \frac{Q_{II}}{m} + \frac{q_{II}}{m}.$$

However, for  $m \geq 2$ , the above inequality leads to

$$\frac{Q_{II}}{m} > q_{II} - \frac{q_{II}}{m} \geq q_{II} - \frac{q_{II}}{2} = \frac{q_{II}}{2}.$$

Combining the above inequality with (3.33), and recalling that  $q_{II} < P$ , leads to

$$\frac{q_{II}}{2} < \frac{Q_{II}}{m} \leq q_{II} < P. \quad (3.34)$$

Expression (3.32) implies that the proof can be completed if we can show

$$\hat{G}(m, Q_{II}) \leq 1.25 [F_{II}(Q_{II}) + H_{II}(q_{II})].$$

However, for this purpose, it is sufficient to prove  $H_{II}(Q_{II}/m) \leq 1.25H_{II}(q_{II})$ . Recall Equation (3.31) which provides an expression for  $H_{II}(Q_b)$ . Also, recall that under the assumptions of this case, the global minimizer of  $H_{II}(Q_b)$ , denoted  $q_{II}$ , is achieved when  $q_{II} < P$ . Then, similarly to Equation (3.5), we can write

$$\frac{H_{II}(Q_b)}{H_{II}(q_{II})} = \frac{1}{2} \left( \frac{Q_b}{q_{II}} + \frac{q_{II}}{Q_b} \right), \quad \forall Q_b \text{ s.t. } 0 \leq Q_b < P.$$

Using the above equation and (3.34), we can write

$$\frac{H_{II}(Q_{II}/m)}{H_{II}(q_{II})} = \frac{1}{2} \left( \frac{Q_{II}/m}{q_{II}} + \frac{q_{II}}{Q_{II}/m} \right) \leq \frac{1}{2} \left( \frac{1}{2} + \frac{2}{1} \right) = 1.25,$$

and this completes the proof for Case 1.

**Case 2:**  $q_{II} < Q_{II}$  and  $q_{II} \geq P$ .

Again, the total cost of the heuristic solution is not greater than the cost of the solution given by  $(m, Q_{II})$ . For this particular case, it is also easy to show that the cost of our heuristic solution is not greater than the cost of the solution given by  $(m, miP)$ .

Let us first analyze the solution given by  $(m, miP)$ . By definition,  $kP \leq Q_{II} < (k+1)P$ , and thus  $k/i \leq Q_{II}/iP < (k+1)/i$ . This in turn implies  $\lfloor Q_{II}/iP \rfloor < (k+1)/i$  so that

$$miP < (k+1)P \leq 2kP \leq 2\sqrt{\frac{2K_v D}{h_v}}.$$

Since  $Q_{II} \geq kP \geq iP \geq P$ , we have

$$miP = \left\lfloor \frac{Q_{II}}{iP} \right\rfloor iP \geq \frac{Q_{II}}{2} > \frac{1}{2}\sqrt{\frac{2K_v D}{h_v}}.$$

It follows that

$$\frac{1}{2}\sqrt{\frac{2K_v D}{h_v}} < miP \leq 2\sqrt{\frac{2K_v D}{h_v}}. \quad (3.35)$$

As a result,  $h(m, miP) = H_{II}(iP) \leq 1.06H_{II}(q_{II})$ . Also, note that

$$\frac{K_v D}{miP} + \frac{h_v miP}{2} = \frac{1}{2} \left( \frac{miP}{\sqrt{2K_v D/h_v}} + \frac{\sqrt{2K_v D/h_v}}{miP} \right) \sqrt{2K_v D h_v}.$$

Then, considering (3.35), we can write

$$\begin{aligned} F_{II}(miP) &\leq \frac{1}{2} \left( \frac{1}{2} + \frac{2}{1} \right) \sqrt{2K_v D h_v} + \frac{RD}{P} \\ &\leq 1.25 \left( \sqrt{2K_v D h_v} + \frac{RD}{P} \right) \leq 1.25 F_{II}(Q_{II}). \end{aligned}$$

Thus,

$$\hat{G}_m(miP) \leq 1.25 F_{II}(miP) + 1.06 H_{II}(q_{II}) \leq 1.25 F_{II}(Q_{II}) + 1.06 H_{II}(q_{II}) \leq 1.25 \hat{G}^*,$$

and this completes the proof for Case 2.

**Case 3:**  $Q_{II} \leq q_{II}$ .

We investigate the objective function over two regions. Namely,  $Q_v < q_{II}$ . and  $Q_v \geq q_{II}$ .

**Case 3.1:**  $Q_v < q_{II}$ .

Over this region, we will show that  $\hat{G}(1, Q_v) \leq \hat{G}(n, Q_v)$ , for all integer  $n \geq 2$ .

We have

$$\hat{G}(1, Q_v) = F_{II}(Q_v) + h(1, Q_v) = F_{II}(Q_v) + H_{II}(Q_v),$$

and

$$\hat{G}(n, Q_v) = F_{II}(Q_v) + H_{II}(Q_v/n).$$

In turn, for our purposes, it suffices to show that  $H_{II}(Q_v) \leq H_{II}(Q_v/n)$ . That is, recalling Expression (3.31), it suffices to show that

$$\frac{K_b D}{Q_v} + \frac{h' Q_v}{2} + \left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} \leq \frac{K_b D}{Q_v/n} + \frac{h' Q_v/n}{2} + \left\lceil \frac{Q_v}{nP} \right\rceil \frac{nRD}{Q_v}.$$

Observe that

$$\left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} \leq \left\lceil \frac{Q_v}{nP} \right\rceil \frac{nRD}{Q_v}.$$

If we consider  $Q_v \leq \sqrt{2K_b D/h'}$  then

$$\frac{K_b D}{Q_v} + \frac{h' Q_v}{2} \leq \frac{K_b D}{Q_v/n} + \frac{h' Q_v/n}{2},$$

and hence  $H_{II}(Q_v) \leq H_{II}(Q_v/n)$  for all  $Q_v \leq \sqrt{2K_b D/h'}$ .

Now, consider  $Q_v > \sqrt{2K_b D/h'}$ . That is, we are considering those  $Q_v$  such that  $\sqrt{2K_b D/h'} < Q_v < q_{II}$ . In this case, it is impossible to have  $q_{II} = kP$  because there does not exist a nonnegative integer  $k$  such that  $kP < \sqrt{2K_b D/h'} \leq (k+1)P$  and  $q_{II} > Q_v$ . It follows that

- $q_{II} = \sqrt{2[K_b + (k+1)R]D/h'}$ , or
- $q_{II} = (k+1)P$ .

First, suppose that  $k = 0$ . That is,  $q_{II} = \sqrt{2(K_b + R)D/h'}$ , or  $q_{II} = P$ . In both cases, we are considering those  $Q_v$  such that  $Q_v/n \leq Q_v < q_{II} \leq P$ . Note that  $H_{II}(Q_v)$  is an EOQ-type function over this range. If  $q_{II} = \sqrt{2(K_b + R)D/h'}$ , then  $H_{II}(Q_v/n) \geq H_{II}(Q_v)$  as  $H_{II}(Q_v)$  is convex over  $(0, P]$  and  $\sqrt{2[K_b + (k+1)R]D/h'}$  is its unique minimizer. If  $q_{II} = P$ , then the point  $\sqrt{2[K_b + R]D/h'}$  is not realizable, i.e.,  $\sqrt{2[K_b + R]D/h'} > P$ . Again, using the EOQ-type properties of  $H_{II}(Q_v)$ , we conclude that  $H_{II}(Q_v/n) \geq H_{II}(Q_v)$ . Now, suppose that  $k \geq 1$ . It follows that  $(k+1)/k \leq 2$ . Thus, over the region  $Q_v < q_{II}$ ,  $Q_v/(kP) \leq 2$  so that  $Q_v/n \leq kP$  for all  $n \geq 2$ . Under these assumptions, it follows from Property 4 that  $H_{II}(Q_v) \geq H_{II}(kP)$

where  $Q_v \leq kP$ , and thus  $H_{II}(Q_v/n) \geq H_{II}(Q_v) \geq H_{II}(kP)$  for all  $n \geq 2$  and  $q_{II} > Q_v > \sqrt{2K_b D/h'}$ . As a result, we have

$$\hat{G}(1, Q_v) \leq \hat{G}_n(n, Q_v) \forall n \geq 2, \quad \forall Q_v < q_{II}.$$

and this completes the proof of Case 3.1.

**Case 3.2:**  $Q_v \geq q_{II}$  and  $q_{II} < P$ .

Over  $Q_v \geq q_{II}$ , we first consider the case where  $q_{II} < P$ . Then, recalling the original assumptions of Case 3, we have  $Q_{II} \leq q_{II} < P$ . It follows from Properties 2 and 4 that  $F_{II}(Q_v) \geq F_{II}(q_{II})$  for all  $Q_v > P$ . Furthermore, since  $q_{II} < P$ , it follows from Algorithm 1 that  $q_{II} = \sqrt{2(K_b + R)D/h'}$ . However, this  $q_{II}$  value is also the minimizer of  $h(n, Q_v)$ , as well as  $H_{II}(\cdot)$ , over the region  $Q_v < P$ . Treating  $n$  as continuous variable and using the convexity of  $h(n, Q_v)$  over  $Q_v < P$ , the first order conditions suggest an  $n$  value of

$$Q_v \sqrt{\frac{h'}{2(K_b + R)D}}.$$

Substituting  $Q_v = \sqrt{2(K_b + R)D/h'}$  in the above, we have  $n = 1$ . As a result, if  $Q_v \geq q_{II}$  and  $q_{II} < P$  then  $\hat{G}(1, q_{II}) = F_{II}(q_{II}) + h(1, q_{II}) \leq F_{II}(Q_v) + h(n, Q_v)$ ,  $\forall n \geq 2$ . Thus, if  $q_{II} < P$  then

$$\min_{Q_v \geq q_{II}} \hat{G}(1, Q_v) = \min_{Q_v \geq q_{II}} \hat{G}_1(Q_v) \leq \hat{G}(1, q_{II}) \leq \min_{Q_v \geq q_{II}} \hat{G}(n, Q_v)$$

**Case 3.3:**  $Q_v \geq q_{II}$  and  $q_{II} \geq P$ .

Suppose that  $q_{II} = kP$  for some integer  $k$ . Then,  $h(1, kP) = H_{II}(q_{II})$ . Furthermore, since we are considering  $Q_v \geq q_{II}$ , we have  $Q_v \geq q_{II} = kP > Q_{II}$ . Over this region,  $F_{II}(kP) \leq F_{II}(Q_v)$  by Property 3. It follows that  $\hat{G}(1, kP) =$

$$F_{II}(kP) + h(1, kP) \leq F_{II}(Q_v) + h(n, Q_v) \leq \hat{G}(n, Q_v), \quad \forall n, Q_v.$$

Now suppose  $q_{II}$  is not an integer multiple of  $P$ . Then it follows from Algorithm 1 that

$$q_{II} = \sqrt{\frac{2[K_b + (k+1)R]D}{h'}},$$

where  $kP < \sqrt{2K_b D}/h' < (k+1)P$ . Hence, we have

$$\frac{1}{2}\sqrt{\frac{2K_b D}{h'}} < \frac{(k+1)P}{2} \leq kP \quad \text{so that} \quad kP < 2\sqrt{\frac{2K_b D}{h'}}.$$

Recalling (3.31), we have

$$h(1, kP) = H_{II}(kP) = \frac{K_b D}{kP} + \frac{h' kP}{2} + \left\lceil \frac{kP}{P} \right\rceil \frac{RD}{kP}.$$

Now, considering the first two terms of the above expression, we analyze the error for replacing the minimizer of these two terms with  $kP$ . Observe that

$$\frac{K_b D}{kP} + \frac{h' kP}{2} = \frac{1}{2} \left( \frac{kP}{\sqrt{\frac{2K_b D}{h'}}} + \frac{\sqrt{\frac{2K_b D}{h'}}}{kP} \right) \sqrt{2K_b D h'}.$$

Observe that the RHS of the above is increasing for  $kP > \sqrt{2K_b D}/h'$  and decreasing for  $kP \leq \sqrt{2K_b D}/h'$ . Therefore, the minimum of the RHS is achieved at either  $kP = 2\sqrt{2K_b D}/h'$  or  $kP = (1/2)\sqrt{2K_b D}/h'$ . In both cases,

$$\left( \frac{kP}{\sqrt{\frac{2K_b D}{h'}}} + \frac{\sqrt{\frac{2K_b D}{h'}}}{kP} \right) \leq 2.5 \quad \text{which implies that} \quad \frac{K_b D}{kP} + \frac{h' kP}{2} = 1.25\sqrt{2K_b D h'}.$$

As a result, we have

$$h(1, kP) = H_{II}(kP) = \frac{K_b D}{kP} + \frac{h' kP}{2} + \left\lceil \frac{kP}{P} \right\rceil \frac{RD}{kP} = 1.25\sqrt{2K_b D h'} + \frac{RD}{P}.$$

As for  $Q_{II}$ , we have two possibilities:  $Q_{II} \leq kP$  or  $Q_{II} > kP$ . If  $Q_{II} \leq kP$  then  $q_{II} > kP \geq Q_{II}$ . Since we are considering  $Q_v \geq q_{II}$ , by Property 3, we have

$F_{II}(kP) \leq F_{II}(Q_v)$  for all  $Q_v$  over the region of interest. On the other hand, if  $Q_{II} > kP$ , then  $kP < Q_{II} \leq q_{II} < (k+1)P$ . In this case, Algorithm 1 implies that function  $F_{II}(Q_v)$  is an EOQ-type convex function over  $kP < Q_v \leq (k+1)P$  with a realizable minimizer given by  $Q_{II}$ . Let us consider a point, say  $kP + \delta$  where  $kP < kP + \delta \leq (k+1)P$  and analyze

$$\frac{F_{II}(kP + \delta)}{F_{II}(Q_{II})} = \frac{1}{2} \left( \frac{kP + \delta}{Q_{II}} + \frac{Q_{II}}{kP + \delta} \right).$$

Observe that the above ratio is increasing for  $kP + \delta > Q_{II}$  and decreasing for  $kP + \delta \leq Q_{II}$ . Therefore, it is minimized at either  $(k+1)P$  or  $\lim_{\delta \rightarrow 0}(kP + \delta)$ . In either case, since  $(k+1)/k \leq 2$ , we have  $(kP + \delta)/Q_{II} \leq 2$ . It follows that

$$\frac{F_{II}(kP + \delta)}{F_{II}(Q_{II})} \leq 1.25 F_{II}(Q_v),$$

and we can write

$$\hat{G}(1, kP) = F_{II}(kP) + h_{II}(1, kP) \leq 1.25 (F_{II}(Q_v) + h_{II}(n, Q_v)) = \hat{G}(n, Q_v).$$

Therefore,

$$\min_{Q_v} \hat{G}(1, Q_v) = \min_{Q_v} \hat{G}_1(Q_v) \leq 1.25 \min_{Q_v} \hat{G}(n, Q_v), \quad \forall n.$$

and this completes the proof. ■

**COROLLARY 3**  $F_{II}(Q_{II}) + H_{II}(q_{II}) \leq \hat{G}^* \leq \bar{G}_{II} \leq 1.25 [F_{II}(Q_{II}) + H_{II}(q_{II})]$ .

**Proof:** The corollary follows from Theorem 3 and Expression (3.32). ■

### III.3.2. Exact Solution for Model II

In this section, we provide an upper bound, denoted  $\hat{n}_{max}$ , on the optimal value of  $n$  for Model II, and develop a finite time exact algorithm. To this end, we utilize

Corollary 3 which gives an upper bound on the optimal cost of Model II.

Let us define  $\hat{g}(n, Q_v)$  as follows:

$$\hat{g}(n, Q_v) = \frac{(K_v + nK_b)D}{Q_v} + \left(h_v + \frac{h'}{n}\right) \frac{Q_v}{2} + \frac{2RD}{P}.$$

Recalling Equation (3.25), observe that

$$\hat{G}(n, Q_v) \geq \hat{g}(n, Q_v), \quad \forall n, Q_v. \quad (3.36)$$

That is, function  $\hat{g}(n, Q_v)$  is, in fact, a lower bound on the objective function of Model II. Let  $\hat{g}^* = \min_{n, Q_v} \hat{g}(n, Q_v)$ , i.e.,  $\hat{g}^*$  denotes the minimum value that  $\hat{g}(n, Q_v)$  can take. It follows from Corollary 3 and Expression (3.36) that

$$\hat{g}^* \leq \hat{G}^* \leq 1.25 [F_{II}(Q_{II}) + H_{II}(q_{II})]. \quad (3.37)$$

Observe that, for a given  $n$ , function  $\hat{g}(n, Q_v)$  is convex in  $Q_v$  with a minimizer at

$$\hat{q}^*(n) = \sqrt{\frac{2(K_v + nK_b)D}{h_v + h'/n}}.$$

Thus, for any given  $n$ ,

$$\hat{g}(n, \hat{q}^*(n)) = \sqrt{2(K_v + nK_b)(h_v + h'/n)D} + \frac{2DR}{P} \leq g(n, Q_v), \quad \forall Q_v. \quad (3.38)$$

Also, it can be easily shown that  $\hat{g}(n, \hat{q}^*(n))$  is decreasing over  $n \leq \lfloor n_0 \rfloor$  and increasing over  $n > \lfloor n_0 \rfloor$  where we again have  $n_0 = \sqrt{K_v h' / K_b h_v}$  so that Lemma 2 follows.



**LEMMA 2** The optimal  $n$  for Model II lies over  $[\hat{n}_{min}, \hat{n}_{max}]$  where

$$\begin{aligned}\hat{n}_{min} &= \max \left\{ \left\lfloor \frac{\hat{N} - \sqrt{\hat{N}^2 - 4K_v K_b h_v h'}}{2K_b h_v} \right\rfloor, 1 \right\}, \\ \hat{n}_{max} &= \left\lceil \frac{\hat{N} + \sqrt{\hat{N}^2 - 4K_v K_b h_v h'}}{2K_b h_v} \right\rceil, \quad \text{and} \\ \hat{N} &= \frac{(1.25 [F_{II}(Q_{II}) + H_{II}(q_{II})] - 2RD/P)^2}{2D} - K_v h_v - K_b h' .\end{aligned}$$

**Proof:** Combining Expressions (3.37) and (3.38), we conclude that the optimal  $n$  lies where

$$\hat{g}(n, \hat{q}^*(n)) \leq 1.25 [F_I(Q_I) + H_{II}(q_{II})] .$$

Setting the expression of  $\hat{g}(n, \hat{q}^*(n))$ , given by (3.38), equal to  $1.25 [F_I(Q_I) + H_{II}(q_{II})]$  and solving for  $n$ , we can easily complete the proof in a similar fashion to the proof of Lemma 1. ■

#### ALGORITHM 6 – Optimal Algorithm for Model II

For  $n = \hat{n}_{min}, \hat{n}_{min} + 1, \dots, \hat{n}_{max}$ , compute  $\hat{G}_n(Q_v)$  using Expression (3.25). For each  $\hat{G}_n(Q_v)$ , execute Algorithm 4. The  $(n, Q_v)$  pair corresponding to

$$\min \left\{ \min_{Q_v \geq 0} \hat{G}(n, Q_v) : n = \hat{n}_{min}, \dots, \hat{n}_{max} \right\}$$

is the optimal solution for Model II.

**COROLLARY 4** Both  $\hat{n}_{min}$  and  $\hat{n}_{max}$  are finite positive integers, and it follows that Algorithm 6 is a finite time exact algorithm for Model II.

#### III.3.3. Numerical Study for Model II

In Theorem 3, the theoretical error bound of the heuristic developed for Model 2 was shown to be 25%. However, the actual performance of the heuristic is, in fact,

better than this worst case error bound. To test the practical performance of the heuristic, we conducted a computational study including 2187 problem instances. These problems were generated using 3 different settings for each parameter (i.e.  $K_v$ ,  $K_b$ ,  $h_v$ ,  $h_b$ ,  $R$ ,  $P$  and  $D$ ). These settings are  $K_v = 175, 350, 700$ ;  $K_b = 50, 100, 150$ ;  $R = 60, 120, 240$ ;  $P = 5, 10, 20$ ;  $D = 2, 4, 8$ ;  $h_v = 0.5, 1, 2$ ;  $h_b = 4, 8, 16$ . Consequently, the echelon inventory holding cost for the retailer takes 9 different values corresponding to each  $(h_v, h_b)$  pair and these values are 3.5, 7.5, 15.5, 3, 7, 15, 2, 6, 14. Based on the above listed parameter values, we developed a factorial design corresponding to the 2187 parameter settings.

Both the heuristic and the optimal algorithms were coded, and the programs were run on a unix-based system. The running times of both the heuristic and the exact algorithms were on the order of seconds. The upper bound ( $\hat{n}_{max}$ ) to calculate the optimum solution, which was obtained using the results of the heuristic, achieved reasonable values. On average, this upper bound was less than 56, and, in 85% of the test instances, it was less than 100. The maximum value it took was 422 which occurred in only one problem instance.

Analyzing the results for the 2187 problems generated as described above, we observed that the actual error of the heuristic was significantly less than its theoretical upper bound. The average actual error and the maximum actual error obtained were 0.215% and 8.092%, respectively. The heuristic algorithm gave the exact optimal solution in 1443 problem instances. The number of problems that resulted in an error percentage bounded in the intervals  $(0, 1]$ ,  $(1, 2]$ ,  $(2, 3]$ ,  $(3, 4]$ ,  $(5, 6]$ ,  $(6, 7]$  and  $(7, 8]$  were 601, 112, 22, 5, 2, 1 and 1, respectively.

This computational study demonstrated that the actual error of the heuristic algorithm can be much less than 25% in practical instances. Also, this heuristic can be used to obtain an effective upper bound on  $n$  which can be used to solve the

problem optimally in a reasonable amount of time.

### III.4. Summary

This chapter extends the classical buyer-vendor coordination model to consider general replenishment cost structures via incorporating inbound and outbound transportation costs for the vendor. The applications of this problem arise where transportation arrangements to and from the vendor call for full-truck-load (FTL) shipments. Another transportation option is to use a common-carrier, such as UPS or FedEx. Common-carrier transportation is a cost effective alternative for less-than-truckload (LTL) shipments, and this option also offers substantial reductions in freight rates when the replenishment quantity exceeds one of the nominal rate breakpoints. The buyer-vendor coordination problem under common-carriage transportation charges remains an area for future research. Since a typical common-carriage cost function is neither concave nor convex, the optimization problem underlying this generalization is challenging.

The models developed in this chapter account for truck/cargo capacities by considering a cost structure with a fixed cost and a finite capacity per truck/cargo. However, it is worthwhile to emphasize that the analyses here assume an unlimited availability of trucks/cargoes, i.e., no “true” capacity constraints are incorporated. Such capacity constraints have the form

$$\left\lceil \frac{Q_v}{P} \right\rceil \leq C_v \quad \text{and} \quad \left\lceil \frac{Q_v}{nP} \right\rceil \leq C_b,$$

where  $C_v$  and  $C_b$  denote the number of trucks/cargoes available for vendor’s and buyer’s replenishments, respectively. Obviously, the above constraints introduce additional nonlinearities and challenges to the problem. However, these can linearized

in the following fashion:

$$Q_v \leq C_v P \quad \text{and} \quad Q_v \leq nC_b P.$$

With these linear constraints, one can use the exact solution of the unconstrained problem, given by Algorithm 6, as a lower bound to compute the solution of the constrained problem. Extensions with “true” capacity constraints remain an open problem. However, the results of the current chapter provide a foundation for future work in this area.

Another notable generalization is the case where the vendor has multiple options in choosing the transportation mode, e.g., different kinds of transportation equipment or different choices of transportation providers with different costs and cargo capacities. Naturally, further important generalizations also include multi-item, multi-buyer, and multi-vendor problems with different transportation considerations.

Several recent papers investigate the buyer-vendor coordination problem where the vendor’s inventory replenishment/production rate is finite (e.g., Goyal 2000, Goyal and Nebebe 2000, Hill 1997, Hill 1999, Hoque and Goyal 2000, Viswanathan 1998). However, the current literature in this area does not consider the general replenishment cost functions modeled here. In this context, the inbound costs considered in this chapter may also represent capacitated production setups at the vendor. In Chapter 5, we analyze such buyer-vendor production models.

Last, but not least, a practical extension of the buyer-vendor problem with general transportation considerations analyzes the case where the vendor offers quantity discounts as in Banerjee (1986a), Joglekar (1988), Lee and Rosenblatt (1986), Monahan (1984), and the question is how to allocate the cost savings achieved through coordination between the buyer and the vendor. This problem is particularly important in the context of VMI contract design, and it is addressed in the next chapter.

## CHAPTER IV

### CHANNEL COORDINATION FOR THE BUYER-VENDOR PURE INVENTORY PROBLEM WITH DETERMINISTIC AND CONSTANT DEMAND

In this chapter, we study the channel coordination problem for the two models in Chapter III. As we have discussed earlier, these models are based on the classical buyer-vendor coordination problem introduced by Goyal (1976). Model I in Chapter III generalizes Goyal's problem so that the vendor has a stepwise replenishment cost structure,  $C_v(Q)$ , given by Expression (1.1) which includes setup and cargo cost components. Model II considers the same problem when both the vendor and the buyer have such replenishment cost structures given by  $C_v(Q)$  and  $C_b(Q)$  in Expressions (3.1) and (3.2).

In his paper, Goyal applies both centralized and decentralized approaches without consideration of cargo costs. He also proposes a judicious method for allocating the savings from the centralized model to the buyer and the vendor. However, he does not compare the optimal values of the decision variables in the centralized and decentralized models. In Section IV.1 of this chapter, we first revisit Goyal's problem and provide a comparative analysis of his results. We then develop decentralized solutions for Models I and II introduced in Chapter III. We also compare the decentralized and centralized solutions for the two models on a large set of problems. We illustrate that the results of the analysis done in Section IV.1 do not necessarily hold when Goyal's problem is generalized to incorporate cargo costs and capacity.

As a final task in this chapter, we propose ways to coordinate the channel for Models I and II. It is important to note that the method proposed in Goyal (1976) to allocate the savings from the centralized approach is based on the assumption that the vendor and the buyer fully corporate. However, our detailed analysis of the optimal

centralized and decentralized approaches provides insights for designing coordination mechanisms that work for the competitive environment as well.

In the current chapter, we will use the same notation as in Chapter III. In order to differentiate our notation for the decentralized and centralized analysis, we introduce the following additional notation:

- $G_b(Q_b)$ : Buyer's cost function in Model I.
- $\hat{G}_b(Q_b)$ : Buyer's cost function in Model II.
- $G_v(Q_b, n)$ : Vendor's cost function.
- $Q_{d,1}^*$ : Buyer's optimum order quantity in decentralized Model I.
- $Q_{c,1}^*$ : Buyer's optimum order quantity in centralized Model I.
- $Q_{d,2}^*$ : Buyer's optimum order quantity in decentralized Model II.
- $Q_{c,2}^*$ : Buyer's optimum order quantity in centralized Model II.
- $\bar{Q}_d^*$ : Buyer's optimum order quantity in the decentralized model without transportation capacity (i.e., decentralized quantity in Goyal (1976)).
- $\bar{Q}_c^*$ : Buyer's optimum order quantity in the centralized model without transportation capacity (i.e., centralized quantity in Goyal (1976)).
- $n_{d,1}^*$ : Optimum value of  $n$  in decentralized Model I.
- $n_{d,2}^*$ : Optimum value of  $n$  in decentralized Model II.
- $n_{c,1}^*$ : Optimum value of  $n$  in centralized Model I.
- $n_{c,2}^*$ : Optimum value of  $n$  in centralized Model II.
- $\bar{n}_d^*$ : Optimum value of  $n$  in decentralized model of Goyal (1976).
- $\bar{n}_c^*$ : Optimum value of  $n$  in centralized model of Goyal (1976).
- $G_d^*$ : Optimum total costs in decentralized Model I.
- $\hat{G}_d^*$ : Optimum total costs in decentralized Model II.

#### IV.1. A Comparative Analysis of Centralized and Decentralized Models in Goyal (1976)

The decentralized solution for Goyal's problem (Goyal 1976) is given by

$$\bar{n}_d^*(\bar{n}_d^* - 1) \leq \frac{K_v h_b}{K_b h_v} \leq \bar{n}_d^*(\bar{n}_d^* + 1), \quad \text{and} \quad (4.1)$$

$$\bar{Q}_d^* = \sqrt{\frac{2K_b D}{h_b}} \quad (4.2)$$

whereas the centralized solution is given by

$$\bar{n}_c^*(\bar{n}_c^* - 1) \leq \frac{K_v h'}{K_b h_v} \leq \bar{n}_c^*(\bar{n}_c^* + 1) \quad (4.3)$$

$$\bar{Q}_c^* = \sqrt{\frac{2D(K_b + K_v/\bar{n}_c^*)}{\bar{n}_c^* h_v + h'}}. \quad (4.4)$$

**PROPOSITION 2**  $\bar{n}_d^* \geq \bar{n}_c^*$ .

**Proof:** Assume that  $\bar{n}_d^* < \bar{n}_c^*$ . Since  $\bar{n}_d^*$  and  $\bar{n}_c^*$  are both positive integers,  $(\bar{n}_c^* - \bar{n}_d^*) \geq 1$ . Multiplying both sides of this inequality by  $\bar{n}_c^* + \bar{n}_d^*$ , we obtain  $(\bar{n}_c^*)^2 - (\bar{n}_d^*)^2 \geq \bar{n}_c^* + \bar{n}_d^*$  which can also be written as  $\bar{n}_c^*(\bar{n}_c^* - 1) \geq \bar{n}_d^*(\bar{n}_d^* + 1)$ . From Expression (4.1) we have  $\bar{n}_d^*(\bar{n}_d^* + 1) \frac{K_b h_v}{K_v} \geq h_b$ . Therefore,

$$\bar{n}_c^*(\bar{n}_c^* - 1) \frac{K_b h_v}{K_v} \geq \bar{n}_d^*(\bar{n}_d^* + 1) \frac{K_b h_v}{K_v} \geq h_b. \quad (4.5)$$

By definition,  $h_b > h'$  so that  $\bar{n}_c^*(\bar{n}_c^* - 1) \frac{K_b h_v}{K_v} > h'$  which can be rewritten as  $\bar{n}_c^*(\bar{n}_c^* - 1) > \frac{K_v h'}{K_b h_v}$ . However, this contradicts Expression (4.3). Therefore,  $\bar{n}_d^* \geq \bar{n}_c^*$ . ■

**PROPOSITION 3**  $\bar{Q}_c^* \geq \bar{Q}_d^*$ .

**Proof:** Assume that there exists a problem instance for which  $\bar{Q}_c^* < \bar{Q}_d^*$ . Then using Expressions (4.2) and (4.4),

$$\sqrt{\frac{2D(K_b + K_v/\bar{n}_c^*)}{\bar{n}_c^*h_v + h'}} < \sqrt{\frac{2K_bD}{h_b}}. \quad (4.6)$$

Note that in the above inequality  $\bar{n}_c^*$  is the optimum value of  $n$  corresponding to  $\bar{Q}_c^*$  in the centralized model. Expression (4.6) implies the following:

$$K_b h_b + \frac{K_v h_b}{\bar{n}_c^*} < \bar{n}_c^* K_b h_v + K_b h'.$$

Substituting  $h_b - h_v$  for  $h'$  in the above expression, we obtain:

$$\bar{n}_c^* K_b h_v - \frac{K_v h_b}{\bar{n}_c^*} > K_b h_v$$

which reduces to

$$\bar{n}_c^*(\bar{n}_c^* - 1) > \frac{K_v h_b}{K_b h_v}. \quad (4.7)$$

The above expression combined with the fact that  $h_b > h'$  implies that  $\bar{n}_c^*(\bar{n}_c^* - 1) > \frac{K_v h'}{K_b h_v}$ . This again contradicts Expression (4.3). Therefore,  $\bar{Q}_c^* \geq \bar{Q}_d^*$ . ■

The above proposition implies that in Goyal's problem (i.e., when cargo costs and capacity are ignored), the vendor should always encourage the buyer to order more to coordinate the channel. Therefore, the traditional literature concentrates on coordination mechanisms that discourage the buyer from ordering small quantities. However, as we will show in this chapter and Chapter VI, when transportation capacity and costs are incorporated, there are cases where small order quantities are better for coordinating the channel. Therefore, coordination mechanisms that will discourage the buyer from ordering more are needed. Moreover, Proposition 2 indicates that the buyer's decentralized order quantity results in more frequent dispatches from the vendor. However, in subsequent parts of this chapter, we show that when transporta-



tion costs are incorporated into the model, this result also does not necessarily hold. Since the number of dispatches within one vendor replenishment cycle only effects the vendor's order quantity (i.e.,  $Q_v = nQ_b$ ), this result does not have implications for the coordination mechanism. However, it is still an important issue for understanding the behavioral differences of the two modelling approaches. A final note on Propositions 2 and 3 is that they are independent of  $K_v$  and  $K_b$ . This enables us to make a clear comparison of the two cases where transportation costs and capacity are modeled vs. not modeled and to analyze the effect of truck capacities.

#### IV.2. Decentralized Solutions for Model I and Model II

We assume a buyer driven channel, and, therefore, in the decentralized approach, the buyer solves his/her subproblem first. Once the buyer decides on the optimum level of his/her order quantity  $Q_b$ , the vendor's decision problem is to determine the number of dispatches  $n$ . The vendor's resulting replenishment quantity per cycle can be found by the relationship  $Q_v = nQ_b$ . The vendor's optimization problem in Models I and II are the same. Therefore, we will first present a proposition that is useful for solving the vendor's subproblem.

Consider the following function  $\psi(n)$  where  $n$  is a positive integer and  $K$ ,  $R$ ,  $P$ ,  $h$  and  $Q$  are positive real numbers.

$$\psi(n) = \frac{KD}{nQ} + \frac{\lceil \frac{nQ}{P} \rceil RD}{nQ} + \frac{h(n-1)Q}{2} \quad (4.8)$$

The following proposition gives lower and upper bounds on the optimal value of  $n$  which minimizes  $\psi(n)$ , (i.e.,  $n^*$ ).

**PROPOSITION 4** Define  $B = \frac{(K+R)D}{Q} + \frac{hQ}{2}$ . Let

$$n_{min} = \max \left( 1, \left\lfloor \frac{B - \sqrt{B^2 - 2KDh}}{hQ} \right\rfloor \right) \quad (4.9)$$

and

$$n_{max} = \left\lceil \frac{B + \sqrt{B^2 - 2KDh}}{hQ} \right\rceil \quad (4.10)$$

Then  $n_{min} \leq n^* \leq n_{max}$ .

**Proof:** Observe that

$$\psi(n) \geq \phi(n) = \frac{KD}{nQ} + \frac{\frac{nQ}{P}RD}{nQ} + \frac{h(n-1)Q}{2}, \quad \forall n \geq 1$$

Treating  $n$  as a continuous variable, it is easy to show that  $\phi(n)$  is a strictly convex function of  $n$ . Denoting the continuous minimizer of  $\phi(n)$  by  $n_o$ , we simply have  $n_o = \frac{1}{Q} \sqrt{\frac{2KD}{h}}$ .

Defining  $n^*$  as the minimizer of  $\psi(n)$ , we also have  $\psi(1) \geq \psi(n^*)$ . Note that  $\psi(1) = \frac{KD}{Q} + \frac{\lceil \frac{Q}{P} \rceil RD}{Q}$ . Since  $\lceil \frac{Q}{P} \rceil < (\frac{Q}{P} + 1)$ , it follows from  $\psi(1) \geq \psi(n^*)$  that  $\frac{(K+R)D}{Q} + \frac{RD}{P} > \psi(1) \geq \psi(n^*)$ . Let  $A = \frac{(K+R)D}{Q} + \frac{RD}{P}$ . We have  $A > \psi(n^*) > \phi(n^*)$ . Since  $A > \psi(n^*) > \phi(n^*)$  and  $\phi(n)$  is a strictly convex function of  $n$ ,  $\phi(n) = A$  has two roots leading to Expressions (4.9) and (4.10) so that  $n_{min} \leq n^* \leq n_{max}$ . ■

Next, we apply decentralized analysis to Models I and II.

#### IV.2.1. Model I

##### IV.2.1.1. Buyer's Subproblem

In Model I, the buyer does not have transportation costs or capacity considerations. Therefore,

$$G_b(Q_b) = \frac{DK_b}{Q_b} + \frac{h_b Q_b}{2} \quad (4.11)$$

leading to  $Q_{d,1}^* = \sqrt{2K_b D/h_b}$ .

#### IV.2.1.2. Vendor's Subproblem

For a given value of the buyer's order quantity, the vendor's decision problem is to find the optimal number of dispatches within one vendor replenishment cycle, i.e., the optimum value of  $n$ . Therefore, the vendor minimizes the following function

$$G_v(Q_{d,1}^*, n) = \frac{\left(K_v + \left\lceil \frac{nQ_{d,1}^*}{P} \right\rceil R\right) D}{nQ_{d,1}^*} + \frac{h_v(n-1)Q_{d,1}^*}{2} \quad (4.12)$$

over  $n \in Z^+$ . The function given in Expression (4.12) is a piecewise function. Therefore, it is not differentiable. However, it can be minimized by a finite enumeration algorithm on  $n$  that is based on Proposition 4. By setting  $K = K_v$ ,  $Q = Q_{d,1}^*$  and  $h = h_v$ , the optimum value of  $n$  is then given by  $\arg \min\{G_v(Q_{d,1}^*, n) : n = n_{min}, \dots, n_{max}\}$ .

### IV.2.2. Model II

#### IV.2.2.1. Buyer's Subproblem

In Model II, the buyer incurs a cost of  $\$R$  for each cargo with capacity  $P$ . Therefore, the buyer's costs, as a function of his/her order quantity in Model II, is given by

$$\hat{G}_b(Q_b) = \frac{DK_b}{Q_b} + \frac{h_b Q_b}{2} + \frac{D \left\lceil \frac{Q_b}{P} \right\rceil R}{Q_b}. \quad (4.13)$$

This function can be minimized using Algorithm 1 in Chapter III.

#### IV.2.2.2. Vendor's Subproblem

For a given value of the buyer's optimum order quantity, the vendor's subproblem in Model II is the same as in Model I. That is the vendor's total cost function in Model

II is given by

$$G_v(Q_{d,2}^*, n) = \frac{\left(K_v + \left\lceil \frac{nQ_{d,2}^*}{P} \right\rceil R\right) D}{nQ_{d,2}^*} + \frac{h_v(n-1)Q_{d,2}^*}{2}. \quad (4.14)$$

By setting  $K = K_v$ ,  $Q = Q_{d,2}^*$  and  $h = h_v$ , we again have  $n_{d,2}^* = \arg \min\{G_v(Q_{d,2}^*, n) : n = n_{min}, \dots, n_{max}\}$ . The vendor's optimal replenishment quantity in Model II is then equal to  $Q_{d,2}^* n_{d,2}^*$ .

### IV.3. Experimental Analysis

In order to see the impact of the centralized solution on the cost savings and to characterize the cases for channel coordination, we made an extensive numerical analysis. By using the same set of problems described in Chapter III, we solved the centralized and decentralized solutions for Model I and Model II. Based on this sample of problems, we observed improvement as high as 13% in total system costs by coordinating the channel. Furthermore, in contrast to the problem in Goyal's paper, we have obtained results where the following may occur:

1.  $Q_{c,j}^* \geq Q_{d,j}^*$ ,  $n_{c,j}^* \leq n_{d,j}^*$ : This is the same case as in Goyal's problem.
2.  $Q_{c,j}^* < Q_{d,j}^*$ ,  $n_{c,j}^* < n_{d,j}^*$ : Since  $Q_{c,j}^* < Q_{d,j}^*$ , this contradicts Proposition 3.
3.  $Q_{c,j}^* < Q_{d,j}^*$ ,  $n_{c,j}^* > n_{d,j}^*$ : This contradicts both Proposition 3 and Proposition 2.
4.  $Q_{c,j}^* > Q_{d,j}^*$ ,  $n_{c,j}^* > n_{d,j}^*$ : Since  $n_{c,j}^* > n_{d,j}^*$ , this contradicts Proposition 2.

Here,  $j = 1$  for Model I and  $j = 2$  for Model II. Now we will illustrate these cases with some numerical examples.

**Example 1**  $K_v = 175$ ,  $K_b = 50$ ,  $h_v = 2$ ,  $h_r = 4$ ,  $R = 240$ ,  $P = 20$  and  $D = 2$ . In

this instance, we have  $Q_{d,1}^* = 7.071$ ,  $n_{d,1}^* = 5$ ,  $Q_{c,1}^* = 10$ ,  $n_{c,1}^* = 2$  (illustrates the first case).

**Example 2**  $K_v = 350$ ,  $K_b = 150$ ,  $h_v = 0.5$ ,  $h_r = 4$ ,  $R = 240$ ,  $P = 20$  and  $D = 2$ . In this instance, we have  $Q_{d,1}^* = 12.247$ ,  $n_{d,1}^* = 6$ ,  $Q_{c,1}^* = 12$ ,  $n_{c,1}^* = 5$  (illustrates the second case).

**Example 3**  $K_v = 350$ ,  $K_b = 150$ ,  $h_v = 0.5$ ,  $h_r = 4$ ,  $R = 60$ ,  $P = 20$  and  $D = 2$ . In this instance, we have  $Q_{d,1}^* = 12.247$ ,  $n_{d,1}^* = 4$ ,  $Q_{c,1}^* = 12$ ,  $n_{c,1}^* = 5$  (illustrates the third case).

**Example 4**  $K_v = 700$ ,  $K_b = 150$ ,  $h_v = 0.5$ ,  $h_r = 8$ ,  $R = 120$ ,  $P = 10$  and  $D = 2$ . In this instance, we have  $Q_{d,1}^* = 8.66$ ,  $n_{d,1}^* = 8$ ,  $Q_{c,1}^* = 8.889$ ,  $n_{c,1}^* = 9$  (illustrates the fourth case).

#### IV.4. Channel Coordination for Model I and Model II

##### IV.4.1. Coordinated Solution for Model I

In order to develop a coordinated solution for Model I, we consider two cases:  $Q_{d,1}^* > Q_{c,1}^*$  and  $Q_{c,1}^* > Q_{d,1}^*$ . Unlike in Goyal's problem (Goyal 1976), there may be some problem instances where  $Q_{d,1}^* > Q_{c,1}^*$ . However, as a result of our experimental analysis, we have observed that this is not very common in deterministic infinite horizon models.

**PROPOSITION 5** The following mechanisms coordinate the channel for Model I:

- If  $Q_{d,1}^* < Q_{c,1}^*$ , a unit discount of  $\frac{G_b(Q_{c,1}^*) - G_b(Q_{d,1}^*)}{Q_{c,1}^*}$  is offered by the vendor for order sizes greater than or equal to  $Q_{c,1}^*$

- If  $Q_{d,1}^* > Q_{c,1}^*$ , a unit discount of  $\frac{G_b(Q_{c,1}^*) - G_b(Q_{d,1}^*)}{Q_{c,1}^*}$  is offered by the vendor for order sizes less than or equal to  $Q_{c,1}^*$

**Proof:** Note that, in both cases, if the buyer orders  $Q_{c,1}^*$  units, his/her cost is no higher than under the uncoordinated decentralized solution, because his/her costs decrease by  $G_b(Q_{c,1}^*) - G_b(Q_{d,1}^*)$ . Next, we will show that under the new pricing scheme, the buyer's costs are no less than  $G_b(Q_{d,1}^*)$ .

Since the buyer's cost function is a convex function with a minimizer at  $Q_{d,1}^*$ ,  $G_b(Q)$  is strictly increasing for order sizes larger than  $Q_{d,1}^*$ . In addition,  $Q_{d,1}^*$  is independent of the wholesale price. Therefore, if  $Q_{d,1}^* < Q_{c,1}^*$ , under the new pricing scheme, the buyer's cost function attains its minimum at  $Q_{d,1}^*$  for order sizes less than  $Q_{c,1}^*$ . For  $Q \geq Q_{c,1}^*$ , because of the strict convexity of the cost function,  $Q_{c,1}^*$  gives the minimum cost.

For the second part of the proposition (i.e.,  $Q_{d,1}^* > Q_{c,1}^*$ ), we will again use the strict convexity of  $G_b(Q)$  and the fact that  $Q_{d,1}^*$  is its minimizer. For the reasons given,  $G_b(Q)$  is strictly decreasing for order sizes of less than  $Q_{d,1}^*$ . Again the convexity properties of the cost function before  $Q_{c,1}^*$  do not change when the wholesale price changes. Hence  $Q_{c,1}^*$  is the minimizer among the order sizes less than or equal to  $Q_{c,1}^*$ .

■

#### IV.4.2. Coordinated Solution for Model II

**PROPOSITION 6** The following mechanisms coordinate the channel for Model II:

Define  $l_1 = \left\lfloor \frac{Q_{c,2}^*}{P} \right\rfloor$ ,  $l_2 = \left\lceil \frac{Q_{c,2}^*}{P} \right\rceil$  and  $Q_{l_2} = \sqrt{\frac{2(K_b + l_2 R)D}{h_b}}$ . That is,  $Q_{l_2}$  is the economic order quantity when  $l_2$  trucks are used and  $l_2$  is the number of trucks needed by  $Q_{c,2}^*$  units.

- If  $Q_{d,2}^* < Q_{c,2}^*$

- If  $Q_{c,2}^* \geq Q_{l_2}$ , a fixed payment of  $\hat{G}_b(Q_{c,2}^*) - \hat{G}_b(Q_{d,2}^*)$  is paid by the vendor to the buyer for order sizes larger than or equal to  $Q_{c,2}^*$
- If  $Q_{c,2}^* < Q_{l_2}$ , a fixed payment of  $\hat{G}_b(Q_{c,2}^*) - \hat{G}_b(Q_{d,2}^*)$  is paid by the vendor to the buyer for order sizes in the range  $(l_1P, Q_{c,2}^*]$
- If  $Q_{d,2}^* > Q_{c,2}^*$ , a fixed payment of  $\hat{G}_b(Q_{c,2}^*) - \hat{G}_b(Q_{d,2}^*)$  is paid by the vendor to the buyer for order sizes in the range  $(l_1P, Q_{c,2}^*]$

**Proof:** Since the buyer's costs are reduced by  $\hat{G}_b(Q_{c,2}^*) - \hat{G}_b(Q_{d,2}^*)$  if he/she orders  $Q_{c,2}^*$  units, he/she stays in a “no worse” situation by ordering this amount. Next, we show that in each case the buyer's costs are no less than  $\hat{G}_b(Q_{d,2}^*)$ .

If  $Q_{d,2}^* < Q_{c,2}^*$  and  $Q_{c,2}^* \geq Q_{l_2}$ , we have  $\hat{G}_b(l_2P) \geq \hat{G}_b(Q_{c,2}^*)$ . This is because  $Q_{l_2}$  is the economic order quantity when  $l_2$  trucks are used and  $Q_{c,2}^* \geq Q_{l_2}$ . From Property 3 in Chapter III,  $\forall Q > l_2P$  we know that  $\hat{G}_b(Q) > \hat{G}_b(l_2P)$ . Therefore  $\forall Q > Q_{c,2}^*$ ,  $\hat{G}_b(Q) > \hat{G}_b(Q_{c,2}^*)$ .

If  $Q_{d,2}^* < Q_{c,2}^*$  and  $Q_{c,2}^* < Q_{l_2}$ , we know that  $\hat{G}_b(Q)$  is decreasing in the range  $(l_1P, Q_{c,2}^*]$ . Since the change in wholesale price does not effect this property, for this region,  $Q_{c,2}^*$  gives the minimum cost. This is also true when  $Q_{d,2}^* > Q_{c,2}^*$ . ■

#### IV.5. Summary

In this chapter, we studied the channel coordination problem for the two models discussed in Chapter III. In order to illustrate the importance of generalized transportation cost  $C(Q)$ , we first discussed some of the properties of Goyal's model (Goyal 1976), which our two models are based on. Although this is a well-known paper in the literature, to our best knowledge, no other study makes a comparative analysis of the optimum values of the decision variables in the decentralized and centralized models for this problem. We showed that the decentralized model for Goyal's problem

always results in more frequent dispatches to the buyer and smaller order sizes from the buyer than the centralized model.

We next presented decentralized models for the two problems studied in Chapter III. Based on an extensive numerical analysis, we showed that the results of the comparative analysis of Goyal's model (Goyal 1976) do not necessarily hold for these two models. An important implication of this is that, contrary to the common belief in the literature, it is not always best to encourage the buyer to order more to coordinate the channel. For both Model I and Model II, we proposed two coordination mechanisms which rely on these results.

Another generalization of the problems considered in Chapter III and this chapter is the case of stochastic demand, which we consider in Chapter VI.



## CHAPTER V

INTEGRATED BUYER-VENDOR PRODUCTION MODELS WITH  
DETERMINISTIC AND CONSTANT DEMAND

In Section II.3.2 of Chapter II, we provided a review of the existing production/inventory models for buyer-vendor coordination, where the problem is to find the centralized replenishment/dispatch quantities of a buyer-vendor system with a finite production (i.e., replenishment) rate at the vendor. In this chapter, we revisit this class of problems and present a unified centralized model which takes into account the generalized replenishment cost structure of interest in the dissertation. We show that this general formulation can be reduced to the previously studied models for buyer-vendor coordination that suggest various dispatch policies for the vendor. The particular dispatch policies include LFL, IDQ, DWP, F- $\lambda$ , IF- $\lambda$ , 1-unequal, and e-unequal Policies. Using this new formulation, we also prove some optimality properties of the finite production rate problem considering the general replenishment cost structure  $C(Q)$  given by Expression (1.1). Finally, we report the results of an extensive numerical study where we compare these dispatch policies using different problem parameters.

Before presenting the general formulation, we introduce the following additional notation where we define a “production cycle” as the time between two successive production initiations at the vendor.

$\vartheta$ : Vendor’s *annual* production rate.

$q_i$ : Size of the  $i^{th}$  buyer replenishment/dispatch in a production cycle.

$\vec{Q}_b$ :  $\vec{Q}_b = (q_1, q_2, \dots, q_n)$ .

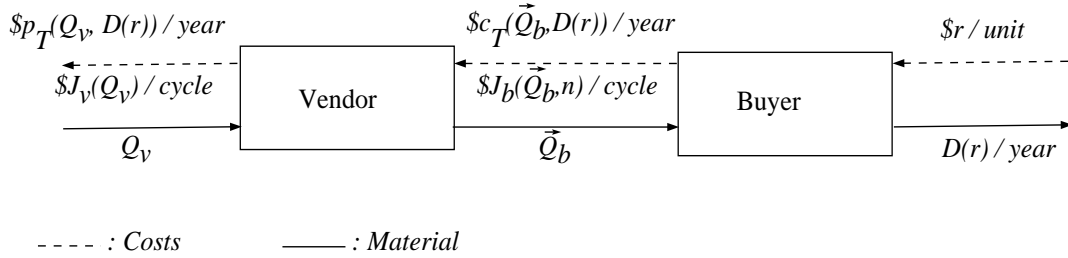
$r$ : Retail price of a unit item.

$D(r)$ : *Annual* demand rate (deterministic, price-sensitive demand).

- $c_T(\vec{Q}_b, D(r))$ : Total *annual* wholesale payment collected by the vendor.  
 $p_T(Q_v, D(r))$ : Vendor's total *annual* material/purchasing cost.  
 $J_b(\vec{Q}_b, n)$ : Buyer's replenishment cost per production cycle.  
 $J_v(Q_v)$ : Vendor's replenishment cost per production cycle.  
 $\alpha$ : Total echelon inventory in the buyer-vendor system when the production of a batch starts.  
 $H_v(Q_v, \vec{Q}_b, n, \alpha)$ : Vendor's *annual* inventory holding cost.  
 $H_b(Q_v, \vec{Q}_b, n)$ : Buyer's *annual* inventory holding cost.  
 $VP(Q_v, \vec{Q}_b, n, \alpha)$ : Vendor's *annual* profit.  
 $BP(Q_v, \vec{Q}_b, n)$ : Buyer's *annual* profit.

Figure 12 shows the cost and quantity flows in a general buyer-vendor system.

Figure 12 Illustration of the Cost and Material Flows



### V.1. A Unified Centralized Model

By considering an infinite horizon setting, the following centralized model maximizes the total annual system profits.

$$\begin{aligned}
& \max \quad VP(Q_v, \vec{Q}_b, n, \alpha) + BP(Q_v, \vec{Q}_b, n) \\
& \text{s.to} \quad Q_v = \sum_{i=1}^n q_i \\
& \quad \frac{1}{D(r)} \left( \alpha + \sum_{j=1}^i q_j \right) \geq \frac{1}{\vartheta} \sum_{j=1}^{i+1} q_j \quad i = 0 \dots n-1 \\
& \quad Q_v \geq 0, \quad n \in Z^+, \quad \alpha \geq 0 \\
& \quad q_i \geq 0 \quad i = 1 \dots n
\end{aligned}$$

where

$$VP(Q_v, \vec{Q}_b, n, \alpha) = c_T(\vec{Q}_b, D(r)) - H_v(Q_v, \vec{Q}_b, n, \alpha) - \frac{J_v(Q_v)D(r)}{Q_v} - p_T(Q_v, D(r)) \quad (5.1)$$

and

$$BP(Q_v, \vec{Q}_b, n) = rD(r) - H_b(Q_v, \vec{Q}_b, n) - \frac{J_b(\vec{Q}_b, n)D(r)}{Q_v} - c_T(\vec{Q}_b, D(r)). \quad (5.2)$$

The above model allows complex pricing schedules for both the vendor's and the buyer's purchasing costs. Representing the total annual purchasing costs as functions of the order quantities and the demand rate, allows us to model situations where the unit purchase price at the buyer, or the material cost for the vendor, depends on the order quantity (e.g., all-unit quantity discounts, incremental quantity discounts). As another generalization, we model demand as a function of retail price. Although we have not considered this situation in our models up to now, in practice, demand is usually a decreasing function of retail price (i.e.,  $r$ ) which can be treated as a decision variable.

We also represent the replenishment costs per production cycle as functions of the

replenishment/dispatch quantities. This allows the incorporation of transportation costs and capacities, or set-up costs due to batch-production, as we explain further in the next section. Finally, we express the annual inventory holding costs as functions of the order quantities. Using this generalization, one can model the complex cases where the unit inventory holding cost depends on the unit purchase price which in turn depends on the order quantity.

As a final explanation about the above model, we note that the second constraint implies the following: In order to be able to deliver the  $(i + 1)^{st}$  dispatch in time, the time to consume  $\alpha$  plus the first  $i$  dispatches at rate  $D$  should be at least as great as the time to produce the first  $(i + 1)$  dispatches at rate  $\vartheta$  (Hill 1999).

## V.2. Generalized Production/Inventory Models

Recall that Hill (1999) solves the finite production rate problem for buyer-vendor coordination without assuming any specific dispatch policy, and, therefore, he develops the optimal policy. In this section, we incorporate the generalized replenishment cost structure  $C(Q)$ , given by Expression (1.1), into the formulation in Hill (1999). We accomplish this by setting

$$\begin{aligned}
 D(r) &= D, \quad c(q, D(r)) = cD, \quad p_T(Q_v, D(r)) = pD, \\
 H_v(Q_v, \vec{Q}_b, n, \alpha) &= h_v \left[ \left( \alpha + \frac{(\vartheta - D)Q_v}{2\vartheta} \right) - \sum_{i=1}^n \frac{q_i^2}{2Q_v} \right], \\
 H_b(Q_v, \vec{Q}_b, n) &= h_b \sum_{i=1}^n \frac{q_i^2}{2Q_v}, \\
 F_b(\vec{Q}_b, n) &= nK_b + \left\lceil \frac{q_1}{P_b} \right\rceil R_b + \left\lceil \frac{q_2}{P_b} \right\rceil R_b + \dots + \left\lceil \frac{q_n}{P_b} \right\rceil R_b, \\
 F_v(Q_v) &= K_v + \left\lceil \frac{Q_v}{P_v} \right\rceil R_v
 \end{aligned}$$

in our unified model formulation.

Note that, in the above,  $c$  and  $p$  represent the unit wholesale price and the unit material/purchasing cost of the vendor, respectively. Although we use them for truck capacities of the buyer and the vendor,  $P_b$  and  $P_v$  can represent the batch sizes in production. In this case,  $R_b$  and  $R_v$  represent the set-up costs incurred per batch by the buyer and the vendor, respectively.

It turns out that, in this case, the maximization problem in our unified model is equivalent to minimizing

$$\frac{(K_v + nK_b)D}{Q_v} + h_v \left( \alpha + \frac{(\vartheta - D)Q_v}{2\vartheta} \right) + (h_b - h_v) \sum_{i=1}^n \frac{q_i^2}{2Q_v} + \sum_{i=1}^n \left\lceil \frac{q_i}{P_b} \right\rceil \frac{R_b D}{Q_v} + \left\lceil \frac{Q_v}{P_v} \right\rceil \frac{R_v D}{Q_v} \quad (5.3)$$

subject to

$$Q_v = \sum_{i=1}^n q_i \quad (5.4)$$

$$\frac{1}{D} \left( \alpha + \sum_{j=1}^i q_j \right) \geq \frac{1}{\vartheta} \sum_{j=1}^{i+1} q_j, \quad i = 0 \dots n-1. \quad (5.5)$$

$$Q_v \geq 0, \quad n \in \mathbb{Z}^+, \quad \alpha \geq 0, \quad q_i \geq 0, \quad i = 1 \dots n \quad (5.6)$$

Note that the formulation provided by Hill (1999) is very similar to the above except that he does not have the last two terms in Expression (5.3). For his problem, Hill (1999) shows that the optimal value of  $\alpha$ , denoted by  $\alpha^*$ , is given by  $q_1 D / \vartheta$ . Next, we prove that when transportation costs and capacities are included, the result is still valid.

**PROPOSITION 7** In the generalized finite production rate model, given by Expressions (5.3)–(5.6),  $\alpha^* = q_1 D / \vartheta$  regardless of the dispatch policy.

**Proof:** For the time being, we fix the production lot-size,  $Q_v$ , and the number of shipments per production run,  $n$ . We focus on determining a pattern of dispatches

for buyer replenishments that minimizes the cumulative holding cost. Without loss of generality, we assume that successive shipments are non-decreasing in size. This is because, the ordering of the shipments affects the total cost only through  $\alpha$ . In Expression (5.3),  $q_i$ 's only appear in the third and fourth terms and notice that changing the order of  $q_i$ 's does not affect these sums. However,  $\alpha$  value in the second term of Expression (5.3), is affected through  $q_n$ . This is because the echelon inventory in the system at the beginning of a new production cycle is determined by how much inventory is left from the previous cycle. Notice also that  $\alpha$  cannot be decreased by switching larger dispatches with earlier smaller dispatches.

Expression (5.5) implies that

$$\alpha \geq \frac{D}{\vartheta} \sum_{j=1}^{i+1} q_j - \sum_{j=1}^i q_j, \quad i = 0 \dots n-1. \quad (5.7)$$

Also, observe that  $\alpha$  should be as small as possible to minimize Expression (5.3). Therefore, in the minimum cost solution, the following should be satisfied.

$$\alpha = \max_{0 \leq i \leq n-1} \left\{ \frac{D}{\vartheta} \sum_{j=1}^{i+1} q_j - \sum_{j=1}^i q_j \right\}$$

We will show that  $\alpha^*$  is given by  $i = 0$  and hence  $\alpha^* = q_1 D / \vartheta$ . Suppose this is not true, so that

$$\max_{0 \leq i \leq n-1} \left\{ \frac{D}{\vartheta} \sum_{j=1}^{i+1} q_j - \sum_{j=1}^i q_j \right\} = \frac{D}{\vartheta} \sum_{j=1}^{l+1} q_j - \sum_{j=1}^l q_j$$

for some  $l > 0$ . Therefore,  $\exists b > 0$  such that

$$\frac{D}{\vartheta} \sum_{j=1}^{l+1} q_j - \sum_{j=1}^l q_j = \frac{Dq_1}{\vartheta} + \frac{Db}{\vartheta}$$

which is equivalent to

$$q_{l+1} = \left( \frac{\vartheta}{D} - 1 \right) \sum_{j=1}^l q_j + b + q_1.$$

Letting  $y = \left( \frac{\vartheta}{D} - 1 \right) \sum_{j=1}^l q_j + b$ , the above equation leads to  $q_{l+1} = y + q_1$  where  $y > 0$ .

Next, we analyze four cases, and using perturbation arguments, we show in each case that  $\alpha^* = q_1 D / \vartheta$ .

**Case 1: Both  $q_{l+1}$  and  $q_1$  are full truck loads.**

Since  $q_{l+1} > P_b$ , increasing  $q_1$  by  $P_b$  and decreasing  $q_{l+1}$  by  $P_b$  does not change  $n$ . The truck costs for the buyer's replenishment do not also change. This is because, increasing  $q_1$  by  $P_b$  and decreasing  $q_{l+1}$  by  $P_b$  results in one more truck in the 1<sup>st</sup> shipment and one less than truck in the  $(l+1)^{st}$  shipment. However, the corresponding inventory holding costs decrease by

$$\frac{h_b - h_v}{2Q_v} \{q_1^2 + q_{l+1}^2 - (q_1 + P_b)^2 - (q_{l+1} - P_b)^2\} = \frac{h_b - h_v}{2Q_v} \{2P_b(q_{l+1} - q_1) - 2P_b^2\}.$$

Since  $q_{l+1}$  and  $q_1$  are both full truck loads and  $q_{l+1} > q_1$ , we have  $q_{l+1} - q_1 \geq P_b$ . Hence, the above equation is at least zero.

**Case 2:  $q_{l+1}$  is full truck load and  $q_1$  is not full truck load.**

Define  $\xi = \left\lceil \frac{q_1}{P_b} \right\rceil P_b - q_1$ . In words,  $\xi$  is the quantity necessary to increase the size of the first dispatch to a full truck load. Since  $q_{l+1} = y + q_1$ , we can write  $k_1 P_b = y + k_2 P_b - \xi$  where  $k_1 = \left\lceil \frac{q_{l+1}}{P_b} \right\rceil$  and  $k_2 = \left\lceil \frac{q_1}{P_b} \right\rceil$ . Noting  $y > 0$ , we have  $k_1 \geq k_2$ . This implies  $y = (k_1 - k_2)P_b + \xi \geq \xi$  so that  $q_{l+1} - q_1 \geq \xi$ . Now, we increase  $q_1$  by  $\xi$  and decrease  $q_{l+1}$  by  $\xi$ . Since  $q_{l+1} - \xi \geq q_1 > 0$ , decreasing  $q_{l+1}$  by  $\xi$  does not lower it to 0, and,

hence, we still have  $n$  dispatches. Note also that the number of trucks used in the  $1^{st}$  and  $(l+1)^{st}$  shipments do not increase. However, the inventory holding costs decrease by

$$\frac{h_b - h_v}{2Q_v} \{q_1^2 + q_{l+1}^2 - (q_1 + \xi)^2 - (q_{l+1} - \xi)^2\} = \frac{h_b - h_v}{2Q_v} \{2\xi(q_{l+1} - q_1) - 2\xi^2\}.$$

Since  $q_{l+1} - q_1 \geq \xi$ , the above decrease is at least zero.

**Case 3:  $q_{l+1}$  is not full truck load and  $q_1$  is full truck load.**

Define  $\xi = q_{l+1} - \left\lfloor \frac{q_{l+1}}{P_b} \right\rfloor P_b$ . In words,  $\xi$  is the quantity necessary to decrease the size of the  $(l+1)^{st}$  dispatch to a full truck load. Since  $q_{l+1} > q_1$  and  $q_1$  is already a full truck load, we know that  $\xi > 0$ . Since  $\left\lfloor \frac{q_{l+1}}{P_b} \right\rfloor P_b \geq q_1$ , we have  $q_{l+1} - q_1 \geq \xi$ . Now, we shall increase  $q_1$  by  $\xi$  and decrease  $q_{l+1}$  by  $\xi$ . Note that since  $q_{l+1} - \xi > 0$ , the number of dispatches remains the same. The total truck costs also do not change. This is because the number of trucks used in the  $(i+1)^{st}$  shipment decreases by one and the number of trucks used in the  $1^{st}$  shipment increases by one. The decrease in the inventory holding costs is given by

$$\frac{h_b - h_v}{2Q_v} \{2\xi(q_{l+1} - q_1) - 2\xi^2\}. \quad (5.8)$$

Since  $q_{l+1} - q_1 \geq \xi$ , the above expression is at least zero.

**Case 4: Neither  $q_{l+1}$ , nor  $q_1$  is a full truck load.**

Define  $\xi_1 = \left\lceil \frac{q_1}{P_b} \right\rceil P_b - q_1$  and  $\xi_2 = q_{l+1} - \left\lfloor \frac{q_{l+1}}{P_b} \right\rfloor P_b$ . Let  $\xi = \min\{\xi_1, \xi_2, y/2\}$  where  $q_{l+1} = y + q_1$ . Again, increasing  $q_1$  by  $\xi$  and decreasing  $q_{l+1}$  by  $\xi$  does not change any costs other than inventory holding costs. The decrease is once again given by Expression (5.8). Since  $q_{l+1} - q_1 > \xi$ , in this case, Expression (5.8) is greater than zero.



In all of the above cases, we have shown that by making appropriate modifications in dispatch sizes, we can improve the solution if  $\alpha^*$  is implied by  $l > 0$ . Therefore,  $i = 0$  and this completes the proof. ■

Next, we show that the generalized finite production rate model, and, hence, the unified model, can be easily reduced to the previously studied models for buyer-vendor coordination that suggest various dispatch policies. We summarize the objective functions of these previous models in Table III. Recall that the dispatch policies assumed in these models are discussed in Chapter II, and they include the following:

- LFL (Banerjee 1986b),
- IDQ (Banerjee and Burton 1994, Lu 1995),
- DWP (Goyal 1995),
- F- $\lambda$  (Hill 1997),
- IF $\lambda$  (Goyal 2000),
- 1-unequal (Goyal and Nebebe 2000),
- $e$ -unequal (Hoque and Goyal 2000).

These cost functions can be obtained by substituting the following in our generalized finite production rate problem formulation:

- LFL:  $n = 1$ ,  $Q_v = q_1$ ,  $P_b = \infty$ ,  $P_v = \infty$ ,
- IDQ:  $q_1 = q_2 = \dots = q_n = Q_v/n$ ,  $P_b = \infty$ ,  $P_v = \infty$ ,
- DWP:  $q_{i+1} = (\vartheta/D)q_i$ ,  $Q_v = q_1(\frac{(\vartheta/D)^n - 1}{\vartheta/D - 1})$ ,  $P_b = \infty$ ,  $P_v = \infty$ ,
- F- $\lambda$ :  $q_{i+1} = \lambda q_i$ ;  $Q_v = q_1(\frac{\lambda^n - 1}{\lambda - 1})$ ,  $P_b = \infty$ ,  $P_v = \infty$ ,

Table III Total Cost Functions for All the Policies

LFL	$\frac{D(K_v+K_b)}{Q_v} + \frac{Q_v}{2}(h_b + \frac{h_v D}{\vartheta})$
IDQ	$\frac{D(nK_b+K_v)}{Q_v} + \frac{h_v Q_v}{2n} \left[ \frac{D}{\vartheta}(2-n) + (n-1) \right] + \frac{h_b Q_v}{2n}$
DWP	$\frac{D(nK_b+K_v)(\vartheta/D-1)}{q_1((\vartheta/D)^n-1)} + \frac{q_1}{2}(h_v D/\vartheta + h_b) \left( \frac{1+(\vartheta/D)^n}{1+\vartheta/D} \right)$
F- $\lambda$	$\frac{D(nK_b+K_v)(\lambda-1)}{q_1(\lambda^n-1)} + h_v \left[ \frac{Dq_1}{\vartheta} + \frac{(\vartheta-D)q_1(\lambda^n-1)}{2\vartheta(\lambda-1)} \right] + (h_b - h_v) \frac{q_1(\lambda^n+1)}{2(\lambda+1)}$
1-uneq	$\frac{D(nK_b+K_v)}{q_1(1+(n-1)\vartheta/D)} + \frac{q_1}{2} \left[ h_v \frac{2D+(\vartheta-D)(1+(n-1)\vartheta/D)}{\vartheta} \right] + \frac{q_1}{2} \left[ (h_b - h_v) \frac{1+(n-1)(\vartheta/D)^2}{1+(n-1)(\vartheta/D)} \right]$
e-uneq	$\frac{D(nK_b+K_v)}{Q_v} + Q_v \left( \frac{Dh_v}{\vartheta f(n,e)} + \frac{(\vartheta-D)h_v}{2\vartheta} + \frac{h_b-h_v}{2} \frac{\frac{(\vartheta/D)^{2e-1}}{[(\vartheta/D)^{e-1}-1]} + (n-e)(\vartheta/D)^{2e-2}}{[\frac{(\vartheta/D)^{e-1}}{(\vartheta/D)-1} + (n-e)(\vartheta/D)^{e-1}]^2} \right)$

$$f(n,e) = \frac{(\vartheta/D)^e-1}{\vartheta/D-1} + (n-e)(\vartheta/D)^{e-1}$$

- 1-unequal:  $q_1, q_2 = \dots = q_n = (\vartheta/D)q_1$ ,  $Q_v = q_1(1 + (n-1)\vartheta/D)$ ,  $P_b = \infty$ ,  $P_v = \infty$ ,
- e-unequal :  $q_1, q_2 = (\vartheta/D)q_1, \dots, q_e = (\vartheta/D)^{e-1}q_1$ ,  $q_{e+1} = (\vartheta/D)^{e-1}q_1, \dots, q_n = (\vartheta/D)^{e-1}q_1$ ;  $Q_v = \frac{(\vartheta/D)^e-1}{\vartheta/D-1}q_1 + (n-e)(\vartheta/D)^{e-1}q_1$ ,  $P_b = \infty$ ,  $P_v = \infty$ .

Note that all of the constraints of the formulation are already satisfied when a certain policy is assumed. We do not show the IF $\lambda$  Policy in the above list, because it is a policy that results from an algorithmic approach that Goyal (2000) uses to update the solution of the F- $\lambda$  Policy (Hill 1997).

It is worth noting that, the LFL Policy is a special case of all of these policies. Therefore, if it is optimal in general, then all of the algorithms result in this solution. Similarly, the 1-unequal Policy is included in the e-unequal Policy. Both of them assume that the size of the equal shipment is  $\vartheta/D$  times the size of the last unequal shipment. Using the same idea, the F- $\lambda$  (Hill 1997) is a more general policy than the IDQ and DWP.

Obviously, generalized transportation cost structure  $C(Q)$  can be easily incorpo-

rated by substituting the finite values of  $P_b$  and  $P_v$ , and the regarding truck costs  $R_b$  and  $R_v$ . Below, we present two production/inventory models with such replenishment cost structures. Note that, from Proposition 7, we already know that  $\alpha^* = q_1 D/\vartheta$  in these models.

**Example 5** Consider the LFL Policy where  $P_b = P_v = P$  and  $R_b = R_v = R$ . The total cost per unit time is then given by

$$\frac{D(K_v + K_b)}{Q_v} + \frac{Q_v}{2} \left( h_b + \frac{h_v D}{\vartheta} \right) + 2 \left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v}.$$

Note that we already have the total cost expression for each policy in Table III under the assumption that  $P_b = \infty$  and  $P_v = \infty$ . These expressions are obtained by taking  $\alpha = q_1 D/\vartheta$  which is still valid in case of transportation capacities and costs. The third term in the above expression is the total truck costs when  $P_b = P_v = P$ ,  $R_b = R_v = R$ ,  $n = 1$  and  $Q_v = q_1$ . Letting  $\bar{K} = (K_v + K_b)$ ,  $\bar{h} = (h_b + \frac{h_v D}{\vartheta})$ , and  $\bar{R} = 2R$ , this expression can be rewritten as

$$\frac{D\bar{K}}{Q_v} + \frac{Q_v}{2} \bar{h} + \left\lceil \frac{Q_v}{P} \right\rceil \frac{\bar{R}D}{Q_v}$$

which is the same as Expression (3.3) and hence can be minimized using Algorithm 1.

**Example 6** Consider the IDQ Policy where  $P_b = P_v = P$  and  $R_b = R_v = R$ . The total cost per unit time is then given by

$$\frac{D(nK_b + K_v)}{Q_v} + \frac{h_v Q_v}{2n} \left[ \frac{D}{\vartheta} (2 - n) + (n - 1) \right] + \frac{h_b Q_v}{2n} + \left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} + \left\lceil \frac{Q_v}{nP} \right\rceil \frac{nRD}{Q_v}.$$

After some algebraic manipulations, the above expression can be rewritten as

$$\frac{D(nK_b + K_v)}{Q_v} + \frac{Q_v}{2} \left[ h_v - \frac{h_v D}{\vartheta} + \frac{1}{n} \left( h_b + \frac{2h_v D}{\vartheta} - h_v \right) \right] + \left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} + \left\lceil \frac{Q_v}{nP} \right\rceil \frac{nRD}{Q_v}$$

which, in turn, is equal to

$$\frac{D(nK_b + K_v)}{Q_v} + \frac{Q_v}{2} \left( \bar{h} + \frac{\bar{h}'}{n} \right) + \left\lceil \frac{Q_v}{P} \right\rceil \frac{RD}{Q_v} + \left\lceil \frac{Q_v}{nP} \right\rceil \frac{nRD}{Q_v} \quad (5.9)$$

where

$$\bar{h} = h_v - \frac{h_v D}{\vartheta}$$

and

$$\bar{h}' = h_b + \frac{2h_v D}{\vartheta} - h_v.$$

Note that Expression (5.9) is the same as Expression (3.25) and can be minimized using Algorithm 6.

**Example 7** Consider the DWP Policy where  $P_b = P_v = P$  and  $R_b = R_v = R$ . The total cost per unit time is then given by

$$\begin{aligned} & \frac{D(nK_b + K_v)(\vartheta/D - 1)}{q_1((\vartheta/D)^n - 1)} + \frac{q_1}{2} (h_v D/\vartheta + h_b) \left( \frac{1 + (\vartheta/D)^n}{1 + \vartheta/D} \right) + \\ & \sum_{k=1}^{k=n} \left\lceil \frac{q_1(\vartheta/D)^{k-1}}{P} \right\rceil \frac{RD(\vartheta/D - 1)}{q_1((\vartheta/D)^n - 1)} + \left\lceil \frac{q_1 \frac{\vartheta/D^n - 1}{\vartheta/D - 1}}{P} \right\rceil \frac{RD(\vartheta/D - 1)}{q_1((\vartheta/D)^n - 1)} \end{aligned}$$

which, in turn, is equal to

$$\frac{DK}{q_1} + \frac{hq_1}{2} + \left\lceil \frac{q_1}{P_1} \right\rceil \frac{\bar{R}D}{q_1} + \left\lceil \frac{q_1}{P_2} \right\rceil \frac{\bar{R}D}{q_1} + \dots + \left\lceil \frac{q_1}{P_n} \right\rceil \frac{\bar{R}D}{q_1} + \left\lceil \frac{q_1}{P_{n+1}} \right\rceil \frac{\bar{R}D}{q_1} \quad (5.10)$$

where

$$\begin{aligned} K &= \frac{(nK_b + K_v)(\vartheta/D - 1)}{((\vartheta/D)^n - 1)}, \\ h &= (h_v D/\vartheta + h_b) \left( \frac{1 + (\vartheta/D)^n}{1 + \vartheta/D} \right), \\ \bar{R} &= \frac{R(\vartheta/D - 1)}{((\vartheta/D)^n - 1)}, \end{aligned}$$

and

$$P_i = P \left( \frac{D}{\vartheta} \right)^{i-1} \quad i = 1, \dots, n, \quad P_{n+1} = \frac{P(\vartheta/D - 1)}{((\vartheta/D)^n - 1)}.$$

For fixed  $n$ , the sum of the first three terms of Expression (5.10) looks like Expression (3.3). However, Expression (5.10) contains some additional  $n$  terms. Therefore, this is a more general cost expression than those we have considered in Chapter III. Considering  $P_1, P_2, \dots, P_{n+1}$  as the capacities of  $n + 1$  different types of trucks with the same cost  $\bar{R}$ , minimizing Expression (5.10) for fixed  $n$  is equivalent to finding the optimum order quantity in an EOQ setting where the replenishment is made with  $n + 1$  equal sized dispatches using different types of trucks in each dispatch. A generalized version of Algorithm 1 is required to solve this problem for fixed  $n$ . Then the optimum value of  $n$  can again be found by enumerating Expression (5.10) for all possible values of  $n$ . In order for this enumeration procedure to be finite, a bound on  $n$  is required.

**Example 8** Consider the F- $\lambda$  Policy where  $P_b = P_v = P$  and  $R_b = R_v = R$ . The total cost per unit time is then given by

$$\begin{aligned} & \frac{D(nK_b + K_v)(\lambda - 1)}{q_1(\lambda^n - 1)} + h_v \left[ \frac{Dq_1}{\vartheta} + \frac{(\vartheta - D)q_1(\lambda^n - 1)}{2\vartheta(\lambda - 1)} \right] + (h_b - h_v) \frac{q_1(\lambda^n + 1)}{2(\lambda + 1)} + \\ & \sum_{k=1}^{k=n} \left[ \frac{q_1 \lambda^{k-1}}{P} \right] \frac{RD(\lambda - 1)}{q_1(\lambda^n - 1)} + \left[ \frac{q_1 \frac{\lambda^n - 1}{\lambda - 1}}{P} \right] \frac{RD(\lambda - 1)}{q_1(\lambda^n - 1)} \end{aligned}$$

which can also be expressed by Expression (5.10) where

$$\begin{aligned} K &= \frac{(nK_b + K_v)(\lambda - 1)}{(\lambda^n - 1)}, \\ h &= 2 \left[ \frac{D}{\vartheta} + \frac{(\vartheta - D)(\lambda^n - 1)}{2\vartheta(\lambda - 1)} \right] + (h_b - h_v) \frac{(\lambda^n + 1)}{(\lambda + 1)} \\ \bar{R} &= \frac{R(\lambda - 1)}{(\lambda^n - 1)}, \end{aligned}$$

and

$$P_i = P \left( \frac{1}{\lambda} \right)^{i-1} \quad i = 1, \dots, n, \quad P_{n+1} = \frac{P(\lambda - 1)}{(\lambda^n - 1)}.$$

As in Example 7, solving this problem for fixed  $n$  is equivalent to finding the optimum order quantity in an EOQ setting where the replenishment is made with  $n + 1$  equal sized dispatches using different types of trucks in each dispatch. Again an upper bound for optimum value of  $n$  is needed to develop a finite time enumeration algorithm.

**Example 9** Consider the 1-uneq Policy where  $P_b = P_v = P$  and  $R_b = R_v = R$ . The total cost per unit time is then given by

$$\begin{aligned} & \frac{D(nK_b + K_v)}{q_1(1 + (n-1)\vartheta/D)} + \frac{q_1}{2} \left[ h_v \frac{2D + (\vartheta - D)(1 + (n-1)\vartheta/D)}{\vartheta} \right] + \\ & \frac{q_1}{2} \left[ (h_b - h_v) \frac{1 + (n-1)(\vartheta/D)^2}{1 + (n-1)(\vartheta/D)} \right] + \frac{\left( \left\lceil \frac{q_1}{P} \right\rceil + (n-1) \left\lceil \frac{\vartheta/D q_1}{P} \right\rceil \right) RD}{q_1(1 + (n-1)\vartheta/D)} + \\ & \left\lceil \frac{q_1(1 + (n-1)\vartheta/D)}{P} \right\rceil \frac{RD}{q_1(1 + (n-1)\vartheta/D)} \end{aligned}$$

which, in turn, is equal to

$$\frac{KD}{q_1} + \frac{hq_1}{2} + \left\lceil \frac{q_1}{P_1} \right\rceil \frac{\bar{R}D}{q_1} + (n-1) \left\lceil \frac{q_1}{P_2} \right\rceil \frac{\bar{R}D}{q_1} + \left\lceil \frac{q_1}{P_3} \right\rceil \frac{\bar{R}D}{q_1} \quad (5.11)$$

where

$$\begin{aligned} K &= \frac{(nK_b + K_v)}{(1 + (n-1)\vartheta/D)}, \\ h &= \left[ h_v \frac{2D + (\vartheta - D)(1 + (n-1)\vartheta/D)}{\vartheta} \right] + \left[ (h_b - h_v) \frac{1 + (n-1)(\vartheta/D)^2}{1 + (n-1)(\vartheta/D)} \right], \\ \bar{R} &= \frac{R}{q_1(1 + (n-1)\vartheta/D)}, \\ P_1 &= P, \quad P_2 = \frac{PD}{\vartheta}, \text{ and } P_3 = \frac{P}{(1 + (n-1)\vartheta/D)}. \end{aligned}$$

Minimizing Expression (5.11) is equivalent to finding the optimum order quantity in an EOQ setting where the replenishment is made with  $n + 1$  equal sized dispatches using three types of trucks with same costs. The first type of truck has a capacity of

$P_1$  units and is used in the first batch, the second type of truck has a capacity of  $P_2$  units and is used in the next  $n - 1$  batches, and the last type of truck has a capacity of  $P_3$  units and is used in the last batch.

**Example 10** Consider the e-uneq Policy where  $P_b = P_v = P$  and  $R_b = R_v = R$ . The total cost per unit time is then given by

$$\begin{aligned} & \frac{D(nK_b + K_v)}{q_1 f(n, e)} + \left\lceil \frac{q_1 f(n, e)}{P} \right\rceil \frac{RD}{q_1 f(n, e)} \\ & + \frac{q_1}{2} \left( \frac{2Dh_v}{\vartheta} + \frac{(\vartheta - D)f(n, e)h_v}{\vartheta} + \frac{h_b - h_v}{2} \frac{\frac{(\vartheta/D)^{2e-1}}{(\vartheta/D)^2-1} + (n-e)(\vartheta/D)^{2e-2}}{f^2(n, e)} \right) \\ & + \sum_{k=1}^{k=e} \left\lceil \frac{q_1 (\vartheta/D)^{k-1}}{P} \right\rceil \frac{RD}{q_1 f(n, e)} + (n-e) \left\lceil \frac{q_1 (\vartheta/D)^{e-1}}{P} \right\rceil \frac{RD}{q_1 f(n, e)} \end{aligned}$$

which, in turn, is equal to

$$\frac{KD}{q_1} + \frac{hq_1}{2} + \sum_{k=1}^{k=e} \left\lceil \frac{q_1}{P_k} \right\rceil \frac{\bar{R}D}{q_1} + (n-e) \left\lceil \frac{q_1}{P_{e+1}} \right\rceil \frac{\bar{R}D}{q_1} + \left\lceil \frac{q_1}{P_{e+2}} \right\rceil \frac{\bar{R}D}{q_1} \quad (5.12)$$

where

$$\begin{aligned} K &= \frac{(nK_b + K_v)}{f(n, e)}, \quad \bar{R} = \frac{R}{f(n, e)}, \\ h &= \frac{2Dh_v}{\vartheta} + \frac{(\vartheta - D)f(n, e)h_v}{\vartheta} + \frac{h_b - h_v}{2} \frac{\frac{(\vartheta/D)^{2e-1}}{(\vartheta/D)^2-1} + (n-e)(\vartheta/D)^{2e-2}}{f^2(n, e)}, \\ P_k &= P \left( \frac{D}{\vartheta} \right)^{k-1} \quad k = 1, \dots, e, \quad P_{e+1} = P \left( \frac{D}{\vartheta} \right)^{e-1}, \text{ and } P_{e+2} = \frac{P}{f(n, e)}. \end{aligned}$$

Minimizing Expression (5.12) for fixed  $n$  is equivalent to finding the optimum order quantity in an EOQ setting where the replenishment is made with  $n + 1$  equal sized dispatches using  $(e + 2)$  types of trucks with same costs. The optimum value of  $n$  can again be found by enumerating Expression (5.12) for all possible values of  $n$ . In order for this enumeration procedure to be finite, an upper bound on  $n$  is required.

### V.3. Experimental Analysis

In this section, we switch our attention to the previously studied models for buyer-vendor coordination that suggest various dispatch policies for a vendor with a finite replenishment rate. In particular, we compare these dispatch policies on a number of problem instances under their original modeling assumptions. However, some constraints specific to some models (e.g. maximum cost a buyer can incur (Lu 1995), capacity constraints on buyer shipments (Hoque and Goyal 2000)) are relaxed to compare all policies on the same basis.

We have generated 140 problems that satisfy the  $K_b/K_v$ ,  $h_b/h_v$  and  $\vartheta/D$  ratios used in an earlier numerical study by Viswanathan (1998). Note that Viswanathan (1998) compares the IDQ and DWP Policies on a set of numerical examples and tabulates the *(cost of the best IDQ Policy)/(cost of the best DWP Policy)* for varying combinations of  $K_b/K_v$ ,  $h_b/h_v$  and  $\vartheta/D$  ratios. To be consistent with the earlier work, we have modified the following example in such a way that the same data set is used:  $\vartheta = 3200$ ,  $D = 1000$ ,  $h_b = 5$ ,  $h_v = 4$ ,  $K_v = 400$ ,  $K_b = 25$ . This base case is studied in various previously published papers (e.g., Goyal 1988, 1995). Since the entire buyer-vendor literature assumes that  $h_b > h_v$  (because of added value to items as they are carried through the supply chain), we have omitted the ratio  $h_b/h_v = 1$  in Viswanathan (1998). The experimental factors that we have considered in our analysis are given in Table IV.

We say that a policy is robust if the corresponding cost function is insensitive to deviations from  $Q_v^*$  (the optimal value of the vendor's production quantity) and  $n^*$  (the optimal value of the number of buyer replenishments). For all policies, we decreased and increased first  $Q_v$  and then  $n$  by 5% and 25%, respectively, around their optimal values, and we recalculated the optimal values of other parameters.



Table IV Experimental Factors Considered in the Analysis

$h_b/h_v$	1.25	1.5	1.75	2	3			5 values
$K_b/K_v$	0.01	0.05	0.1	0.2	0.5	1	2	7 values
$D/\vartheta$	0.2	0.4	0.6	0.8				4 values
Total number of combinations								140

Using the problems generated according to the experimental factors presented in Table IV, we coded each policy (including the optimal solution) using C and ran them in Unix environment. Although the running times of all algorithms are on the order of seconds, we note that Hill's (1997) algorithm, which performs a search over  $\lambda$ , takes a longer time than the others.

In order to compare the policies in terms of their deviation from optimality and to analyze their behaviors under different scenarios, we define the following measure:

$$\% \text{ deviation} = 100 \frac{(\text{cost of the current policy}) - (\text{cost of the optimal solution})}{\text{cost of the optimal solution}}$$

The  $\% \text{ deviation}$ 's of the seven policies are summarized in Tables V–VIII.

We note the following observations based on these tables.

1. For all of the policies, as the  $D/\vartheta$  ratio increases, the deviation from the optimal solution increases.
2. Out of the 140 problems, the IDQ is better than the DWP in 43 instances. As  $h_b/h_v$  increases, the IDQ outperforms the DWP. As  $D/\vartheta$  increases, the DWP is superior to the IDQ.
3. Our results for  $(\text{cost of the best IDQ Policy})/(\text{cost of the best DWP Policy})$

Table V    % *deviation* Values for All the Policies when  $D/\vartheta = 0.2$ 

$h_b/h_v$	$K_b/K_v$	LFL	IDQ	DWP	F- $\lambda$	IF- $\lambda$	1-uneq	e-uneq
1.25	0.01	26.98	2.31	5.38	1.04	0.25	0.2	0.01
1.25	0.05	18.9	3.56	2.41	1.25	0.1	0.1	0.1
1.25	0.1	14.19	3.94	1.36	1.15	0.05	0.04	0.04
1.25	0.2	9.06	3.76	0.11	0.11	0.11	0.11	0.11
1.25	0.5	2.01	2.01	0.11	0.11	0.11	0.11	0.11
1.25	1	0	0	0	0	0	0	0
1.25	2	0	0	0	0	0	0	0
1.5	0.01	34.71	1.71	11.79	1	0.12	0.08	0.06
1.5	0.05	23.72	2.49	6.56	1.30	0.04	0.03	0.03
1.5	0.1	17.23	2.52	4.06	0.98	0.03	0.02	0.02
1.5	0.2	10.37	2.22	1.31	0.58	1.31	0.02	0.02
1.5	0.5	2.65	1.64	0.73	0	0.73	0.73	0.73
1.5	1	0	0	0	0	0	0	0
1.5	2	0	0	0	0	0	0	0
1.75	0.01	42.05	1.39	17.88	0.93	0.1	0.05	0.05
1.75	0.05	28.58	1.96	10.76	1.13	0.01	0.01	0.01
1.75	0.1	20.94	1.89	7.35	0.97	0.01	0.01	0.01
1.75	0.2	12.88	1.54	3.62	0.53	0.21	0.2	0.2
1.75	0.5	4.15	0.98	2.2	0	2.22	2.2	0.98
1.75	1	0	0	0	0	0	0	0
1.75	2	0	0	0	0	0	0	0
2	0.01	48.92	1.18	23.58	0.84	0.05	0.03	0.03
2	0.05	33.19	1.57	14.72	0.99	0.01	0	0
2	0.1	24.23	1.33	10.27	0.65	0.09	0.08	0.08
2	0.2	15.35	1.07	5.89	0.35	0.72	0.72	0.72
2	0.5	5.59	0.67	3.61	0	3.68	3.07	0.67
2	1	0	0	0	0	0	0	0
2	2	0	0	0	0	0	0	0
3	0.01	72.73	0.76	43.25	0.62	0.01	0.01	0.01
3	0.05	48.90	0.81	28.26	0.57	0.01	0.01	0.01
3	0.1	36.35	0.7	21.03	0.42	0.59	0.2	0.2
3	0.2	24.11	0.54	13.92	0.24	1.22	1.21	0.54
3	0.5	10	0.42	7.95	0.12	2.92	2.88	0.42
3	1	3.54	0.25	3.54	0	8.07	6.07	0.25
3	2	0	0	0	0	0	0	0

Table VI    % *deviation* Values for All the Policies when  $D/\vartheta = 0.4$ 

$h_b/h_v$	$K_b/K_v$	LFL	IDQ	DWP	F- $\lambda$	IF- $\lambda$	1-uneq	e-uneq
1.25	0.01	54.36	4.88	3.95	1.56	0.51	1.99	0.1
1.25	0.05	41.16	7.64	1.35	1.34	0.07	2.19	0.23
1.25	0.1	33.76	9.1	1.35	1.34	1.03	2.58	0.61
1.25	0.2	24.09	8.72	0.01	0.01	0.01	1.44	0.01
1.25	0.5	12.57	7.33	0	0	0	0	0
1.25	1	6.13	6.13	0	0	0	0	0
1.25	2	0.69	0.69	0	0	0	0	0
1.5	0.01	61.75	3.76	8.93	1.68	0.78	1.23	0.01
1.5	0.05	45.52	6.11	4.48	2.23	0.78	1.58	0.27
1.5	0.1	35.34	6.32	2.54	1.54	0.15	0.96	0.06
1.5	0.2	24.93	6.38	0.69	0.67	0.38	0.42	0.32
1.5	0.5	14.34	7.09	1.57	1.57	1.57	1.57	1.57
1.5	1	7.8	7.09	1.57	1.57	1.57	1.57	1.57
1.5	2	2.27	2.27	1.57	1.57	1.57	1.57	1.57
1.75	0.01	69.06	3.15	13.85	1.67	0.91	0.92	0
1.75	0.05	49.53	4.8	7.37	2.15	0.92	0.98	0.07
1.75	0.1	38.37	5.12	4.85	1.9	0.52	0.63	0.06
1.75	0.2	26.76	5.22	2.17	1.56	0.41	0.4	0.4
1.75	0.5	12.93	4.28	0.32	0.32	0.32	0.15	0.15
1.75	1	6.31	4.12	0.16	0.16	0.16	0.16	0.16
1.75	2	0.85	0.85	0.16	0.16	0.16	0.16	0.16
2	0.01	76.04	2.75	18.54	1.61	0.77	0.76	0.03
2	0.05	53.65	4.02	10.32	2.01	0.62	0.61	0
2	0.1	41.30	4.36	7.07	1.87	0.36	0.35	0.18
2	0.2	28.85	4.38	3.85	1.68	0.20	0.19	0.19
2	0.5	14.76	4.76	1.95	1.87	0.69	0.68	0.68
2	1	6.15	2.78	0.01	0.00	0.01	0.01	0.01
2	2	0.7	0.7	0.01	0.00	0.01	0.01	0.01
3	0.01	100.5	1.93	35.04	1.39	0.48	0.47	0.06
3	0.05	68.64	2.55	21.09	1.64	0.22	0.21	0.09
3	0.1	52.30	2.68	15.40	1.62	0.13	0.12	0.12
3	0.2	36.43	2.88	9.96	1.69	0.30	0.30	0.30
3	0.5	17.40	1.79	4.29	0.58	0.11	0.11	0.11
3	1	7.54	1.02	1.32	0.00	1.33	0.45	0.45
3	2	2.02	1.02	1.32	0.00	1.33	1.32	1.02

Table VII    % *deviation* Values for All the Policies when  $D/\vartheta = 0.6$ 

$h_b/h_v$	$K_b/K_v$	LFL	IDQ	DWP	F- $\lambda$	IF- $\lambda$	1-uneq	e-uneq
1.25	0.01	95.68	7.78	2.67	1.75	0.37	4.79	0.16
1.25	0.05	73.22	12.10	0.97	0.98	0.97	6.66	0.41
1.25	0.1	60.90	14.29	1.56	1.56	1.56	7.72	1.43
1.25	0.2	46.29	15.02	2.05	2.06	2.06	7.40	2.05
1.25	0.5	30.84	16.72	5.19	5.19	5.19	8.57	5.19
1.25	1	19.44	14.08	5.19	5.19	5.19	6.62	5.19
1.25	2	10.58	10.58	4.05	4.06	4.05	4.05	4.05
1.5	0.01	102.8	6.21	6.39	2.15	0.67	3.57	0.04
1.5	0.05	76.5	9.83	2.89	2.39	0.73	5.10	0.45
1.5	0.1	60.99	10.73	1.61	1.61	1.60	4.99	0.57
1.5	0.2	46.90	12.58	2.48	2.48	2.48	6.05	1.99
1.5	0.5	27.22	11.43	2.27	2.27	2.27	4.58	2.27
1.5	1	14.48	8.17	0.81	0.82	0.82	2.19	0.81
1.5	2	6.27	5.85	0.00	0.00	0.00	0.00	0.00
1.75	0.01	110.3	5.41	10.36	2.38	1.07	3.03	0.11
1.75	0.05	80.13	8.33	5.00	2.90	0.96	4.11	0.47
1.75	0.1	62.95	8.85	2.85	2.24	0.51	3.89	0.38
1.75	0.2	47.39	10.33	2.82	2.76	1.69	4.67	1.62
1.75	0.5	25.51	8.31	0.90	0.90	0.90	2.41	0.88
1.75	1	17.49	10.07	3.46	3.47	3.46	4.88	3.46
1.75	2	11.46	10.07	4.88	4.88	4.88	4.88	4.88
2	0.01	117.5	4.81	14.13	2.41	1.22	2.61	0.13
2	0.05	83.44	7.07	6.94	2.84	0.98	3.25	0.30
2	0.1	65.93	8.12	4.74	2.99	1.08	3.54	0.68
2	0.2	47.80	8.46	3.11	2.57	1.04	3.34	1.20
2	0.5	25.40	6.89	0.81	0.81	0.81	1.70	0.21
2	1	14.59	6.60	0.91	0.92	0.91	1.80	0.91
2	2	8.71	6.60	2.29	2.30	2.29	2.29	2.29
3	0.01	142.8	3.30	27.38	2.09	1.33	1.61	0.01
3	0.05	96.97	4.61	14.82	2.55	1.34	1.79	0.06
3	0.1	74.69	5.18	10.24	2.79	1.47	1.94	0.51
3	0.2	52.00	4.79	6.04	2.08	0.75	1.20	0.13
3	0.5	30.01	6.15	4.51	3.25	2.15	2.48	2.24
3	1	14.28	4.96	2.40	2.16	1.50	1.64	1.64
3	2	7.04	3.00	0.72	0.72	0.72	0.72	0.72

Table VIII    % *deviation* Values for All the Policies when  $D/\vartheta = 0.8$ 

$h_b/h_v$	$K_b/K_v$	LFL	IDQ	DWP	F- $\lambda$	IF- $\lambda$	1-uneq	e-uneq
1.25	0.01	176.9	12.25	1.59	1.58	1.06	9.95	0.44
1.25	0.05	130.4	17.99	1.55	1.55	1.55	13.97	1.49
1.25	0.1	105.1	19.84	2.67	2.67	2.67	15.03	2.67
1.25	0.2	78.75	20.30	3.89	3.89	3.89	14.97	3.89
1.25	0.5	51.73	21.99	8.02	8.02	8.02	16.08	8.02
1.25	1	35.72	21.14	10.11	10.11	10.11	15.42	10.11
1.25	2	23.89	18.48	10.52	10.52	10.52	13.79	10.52
1.5	0.01	183.5	10.14	4.00	2.65	0.66	8.06	0.37
1.5	0.05	132.2	15.26	2.36	2.36	2.36	1.68	1.63
1.5	0.1	105.4	16.91	2.81	2.81	2.81	12.67	2.64
1.5	0.2	77.73	17.19	3.30	3.30	3.30	12.41	3.30
1.5	0.5	48.64	17.88	5.82	5.82	5.82	12.77	5.82
1.5	1	31.08	15.96	6.35	6.35	6.35	11.06	6.35
1.5	2	21.09	15.24	8.02	8.02	8.02	10.81	8.02
1.75	0.01	190.8	8.83	6.69	3.09	1.02	6.92	0.36
1.75	0.05	133.5	12.76	2.95	2.91	1.91	9.54	1.14
1.75	0.1	105.5	14.35	2.87	2.87	2.87	10.54	2.17
1.75	0.2	76.80	14.51	2.76	2.76	2.76	10.26	2.62
1.75	0.5	48.97	16.65	6.06	6.06	6.06	12.00	6.06
1.75	1	33.25	17.02	8.11	8.11	8.11	12.41	8.11
1.75	2	21.64	15.31	8.51	8.51	8.51	10.98	8.51
2	0.01	197.8	7.75	9.25	3.14	1.18	5.98	0.23
2	0.05	136.7	11.49	4.31	3.67	1.45	8.53	1.29
2	0.1	106.8	12.82	3.55	3.54	3.07	9.33	2.19
2	0.2	76.92	12.76	2.83	2.83	2.83	8.90	2.39
2	0.5	44.45	11.89	2.84	2.84	2.84	7.78	2.84
2	1	28.41	11.86	4.18	4.18	4.18	7.77	4.18
2	2	19.32	12.50	6.45	6.45	6.45	8.59	6.45
3	0.01	224.8	5.63	19.17	3.17	1.78	4.22	0.27
3	0.05	147.5	7.98	9.09	4.02	2.06	5.73	1.07
3	0.1	112.2	8.69	6.22	4.11	2.11	6.08	1.59
3	0.2	78.29	8.22	3.63	3.14	1.51	5.35	1.33
3	0.5	45.47	8.99	3.56	3.56	3.56	5.95	2.91
3	1	26.54	7.64	2.66	2.66	2.66	4.72	2.51
3	2	17.15	8.62	4.51	4.51	4.51	5.89	4.51

ratios are consistent with those of Viswanathan (1998). Furthermore, we observe that in problems where this ratio is 1, the optimal policy is the LFL. As discussed by Viswanathan (1998), by knowing only the  $h_b/h_v$ ,  $D/\vartheta$  and  $K_b/K_v$  values, one can calculate  $(\text{cost of the best IDQ Policy})/(\text{cost of the best DWP Policy})$ . Therefore, without solving the problem, one can check whether or not this ratio is 1 in order to determine if the LFL is the optimal policy for a problem instance.

4. The superior one of all the policies is the  $e$ -unequal Policy introduced by Hoque and Goyal (2000). Out of 140 problems, the cost provided by this policy is the closest to that of the optimal policy in 127 instances. In the remaining 13 problems, either the Factor- $\lambda$  (Hill 1997) or the IF- $\lambda$  (Goyal 2000) performs better than the  $e$ -unequal Policy.
5. The IDQ Policy and the 1-unequal Policy are simpler to implement than the other policies. However, as pointed out in 2) above, the DWP's performance in general is superior to the IDQ. When we compare these three policies, i.e., 1-unequal, DWP and IDQ, they are the absolute winners in 55, 52 and 6 cases, respectively. In 18 problem instances, the DWP and the 1-unequal tie, and, for the rest of the problems, all three policies result in the same percentage deviation from the optimal. Note that at first glance the IDQ Policy may seem a special case of the 1-unequal Policy, but as the results indicate, it is not (i.e. in 6 cases the IDQ is better than the 1-unequal). The reason is that the 1-unequal Policy assumes a strictly smaller shipment size  $q_1$ , followed by shipment sizes given by  $(\vartheta/D)q_1$ . Hence, the 1-unequal Policy is in general better than the IDQ. Sending an initial smaller shipment can be thought of as updating the IDQ.
6. The increase in % deviation for the 1-unequal Policy, with respect to the  $D/\vartheta$

ratio, is much steeper for larger values of  $D/\vartheta$  than in the other policies. Especially when  $D/\vartheta = 0.8$ , this policy proves inferior. The reason is that as  $D/\vartheta$  increases (i.e.,  $\vartheta/D$  decreases), the size of the first shipment increases drastically, and this quantity is critical in this policy. Such an increase in the size of first shipment results in higher inventory holding costs for the buyer, and, hence higher total costs.

7. In some cases, the IF- $\lambda$  (Goyal 2000) results in higher costs than F- $\lambda$  (Hill 1997). One example is the case where  $h_b/h_v = 1.5$ ,  $K_b/K_v = 0.2$ ,  $D/\vartheta = 0.2$ .
8. When  $D/\vartheta = 0.2$  and  $K_b/K_v = 2$ , the LFL is the superior policy. Except for the case when  $h_b/h_v = 3$ , the LFL is also an optimal solution for  $D/\vartheta = 0.2$  and  $K_b/K_v = 1$ . This is because, if the buyer's replenishment cost increases, for a fixed production batch size at the vendor, it is better to make larger shipments (smaller number of replenishments) to the buyer. However this may result an increase in inventory holding costs for the buyer. This tradeoff is one reason why the LFL is not optimal. On the other hand, for larger values of  $D/\vartheta$ , the frequency of optimality with the LFL diminishes. This is due to the fact that as  $D/\vartheta$  increases, the production rate decreases with respect to the demand rate. However, no backordering is allowed in any case. Therefore, satisfying demand while waiting for large quantities to accumulate (as in Lot-for-Lot) becomes more difficult. Hence Lot-for-Lot may not be optimal in these cases.

#### V.4. Summary

In this chapter, we presented a unified model for deterministic, infinite horizon problems. This general model can be used to incorporate the following issues discussed in the literature and within this dissertation: demand and retail price dependency, order

quantity and wholesale price dependency, and transportation capacities and costs. We proved a very important property which is essential in modeling and solving centralized production/inventory problems with generalized replenishment costs. By using this property, we illustrated that the algorithms we developed to solve Models I and II in Chapter III can also serve as the basis for solving generalized production/inventory models. We also compared the performances of different dispatch policies using an extensive numerical analysis.

This is the final chapter in which we consider deterministic demand, infinite horizon buyer-vendor coordination problems. In the following parts of the dissertation, we analyze the coordination problem under the stochastic demand assumption in different settings.



## CHAPTER VI

SINGLE PERIOD STOCHASTIC DEMAND PROBLEM WITH GENERALIZED  
REPLENISHMENT COSTS

In this chapter, we study the effects of the generalized replenishment cost structure  $C(Q)$ , given by Equation (1.1) on page 4, on the coordination problem of a buyer-vendor pair operating in a Newsboy setting. That is, the buyer faces stochastic demand for a single period, and before the period begins, he/she has a single opportunity to replenish from an outside supplier, i.e. vendor. In computing the buyer's replenishment quantity, we should consider the stochastic nature of the demand (whose distribution is known and density function is denoted by  $f(x)$ ), the possible costs associated with stockouts and excess inventory once the period is over, and the generalized replenishment costs associated with receiving the replenishment order. Since the problem is nonrecurring, we consider the case where the vendor's inbound replenishment quantity is equal to its outbound shipment quantity, which is specified by the buyer's replenishment quantity. Thus, in computing this common replenishment quantity, we also should take into account the vendor's generalized replenishment costs. This chapter develops centralized and decentralized models for computing the replenishment quantity and addresses channel coordination issues in this context.

More specifically, we show that vendor's profit is not always an increasing function of buyer's order quantity and that there are cases where the vendor, in fact, desires a smaller order quantity from the buyer. In these cases, unit discounts are still a valid negotiation mechanism for channel coordination except they play a different role. That is, rather than acting as a motivation for the buyer to increase his/her order quantity, unit discounts aim to compensate the buyer for his losses as a result

of increasing or decreasing his/her order quantity.

As in Chapter III, we develop two models. Model 1 is a special case where the generalized replenishment cost structure is incorporated into the vendor's costs only. Model 2 is a generalization of Model 1, which considers the generalized replenishment cost structure for both the vendor and the buyer. Before going into details of these models, we provide a more detailed discussion of the operational characteristics of the system in our models in this chapter as well as the notation used.

### VI.1. Problem Definition

Depending on the available supply (replenishment quantity) at the buyer, either one of the following cases arise. If demand during the period exceeds the supply, then the buyer is out of stock and additional demand is lost with a  $\$b/unit$  lost sale cost. On the other hand, if demand during the period is less than the available supply at the buyer, then there are excess items at the buyer which can be sold at a salvage value of  $\$v/unit$ . The retail price at the buyer is fixed at  $\$r/unit$ .

The vendor simply orders or produces the buyer's required order quantity. It is worthwhile to note that the general replenishment cost structure of the vendor can also be considered as a capacitated set-up cost due to production at the vendor's site in response to the buyer's order. Therefore, the results presented here are potentially applicable to the case where the vendor is a make-to-order manufacturer (i.e., lot-for-lot manufacturer). The vendor incurs a  $\$p/unit$  production, or purchase cost, and charges a unit wholesale price of  $\$c/unit$  ( $v < p < c < r$ ). As in Chapter III, in addition to a fixed replenishment cost denoted by  $K_v$ , the vendor incurs a transportation cost, given by  $\lceil Q/P_v \rceil R_v$  for an order quantity of  $Q$  units, where  $P_v$  is the truck capacity and  $R_v$  is the per truck cost. As we noted earlier, in Model II

we model the generalized replenishment cost structure for the buyer as well as the vendor. We denote the buyer's truck capacity and cost by  $P_b$  and  $R_b$ , respectively. The buyer's fixed cost of replenishment is denoted by  $\$K_b$  in both models.

Note that the notation used in this chapter is based on the notation used in Chapters III and IV with some modifications. In the models studied here, the vendor's and buyer's replenishment quantities are the same. Therefore, we do not use a separate decision variable for the vendor's order quantity. Additionally, we consider the more general case where the two parties may have different per truck costs and capacities. We index these parameters to make this distinction.

Next, we provide a summary of the additional notation introduced so far as well as a list of some new notation that will be used throughout the chapter.

- $Q$ : Number of items ordered by the buyer.
- $X$ : Random variable showing total demand at the buyer.
- $F(\cdot)$ : Distribution function of demand.
- $p$ : Vendor's procurement cost/unit.
- $v$ : Salvage value/unit at the buyer.
- $c$ : Wholesale price.
- $r$ : Retail price.
- $\bar{\Pi}_b(Q)$ : Buyer's expected profit function excluding truck costs.
- $\bar{\Pi}_v(Q)$ : Vendor's expected profit function excluding truck costs.
- $\bar{\Pi}_c(Q)$ : Expected system profit function excluding truck costs  
(i.e.,  $\bar{\Pi}_c(Q) = \bar{\Pi}_v(Q) + \bar{\Pi}_b(Q)$ ).
- $\Pi_b(Q)$ : Buyer's expected profit function with truck costs.
- $\Pi_v(Q)$ : Vendor's expected profit function with truck costs.

$\Pi_c^I(Q)$ : Expected system profit function for Model I

(i.e.,  $\Pi_c^I(Q) = \Pi_v(Q) + \bar{\Pi}_b(Q)$ ).

$\Pi_c^{II}(Q)$ : Expected system profit function for Model II

(i.e.,  $\Pi_c^{II}(Q) = \Pi_v(Q) + \Pi_b(Q)$ ).

$f(\cdot)$ : Probability density function of demand.

$F(\cdot)$ : Probability distribution function of demand.

$R_b$ : Buyer's cost per truck.

$R_v$ : Vendor's cost per truck.

$P_b$ : Truck capacity in buyer's replenishment.

$P_v$ : Truck capacity in vendor's replenishment.

The buyer's expected profit function, without truck costs ( $\bar{\Pi}_b(Q)$ ), can be found in the Newsboy Problem section of any production and inventory control book. For an extensive discussion of this problem, we suggest Silver et al. (1998, pp. 385–392). Using the notation defined above,  $\bar{\Pi}_b(Q)$  is given by

$$\begin{aligned} \bar{\Pi}_b(Q) = & (-c + v)Q - K_b + (r - v) \int_0^\infty xf(x)dx + \\ & -(r - v + b) \int_Q^\infty (x - Q)f(x)dx \end{aligned} \quad (6.1)$$

$\bar{\Pi}_b(Q)$  is a strictly concave function of  $Q$  with a maximizer that satisfies the following equation.

$$F(\bar{Q}_d^*) = \frac{r + b - c}{r + b - v} \quad (6.2)$$

Here,  $\bar{Q}_d^*$  is the optimal value of the buyer's order quantity in the decentralized system if truck costs for the buyer are ignored. Note that the total cost associated with each demand that cannot be met is  $r + b - c$ , and, hence, the underage cost  $c_u$  is equal to  $r + b - c$ . Similarly  $c - v$  is the cost of each item that is not sold. Therefore, the

overage cost,  $c_o$ , is given by  $c - v$ . By this interpretation, Expression (6.2) can be rewritten in terms of  $c_u$  and  $c_o$  as follows:

$$F(\bar{Q}_d^*) = \frac{c_u}{c_o + c_u}.$$

When truck costs are excluded, the vendor's profits as a function of the buyer's order quantity (i.e.  $\bar{\Pi}_v(Q)$ ) is  $(c - p)Q - K_v$ . It follows that

$$\begin{aligned} \bar{\Pi}_c(Q) = & (-p + v)Q - K_b - K_v + (r - v) \int_0^\infty x f(x) dx + \\ & -(r - v + b) \int_Q^\infty (x - Q) f(x) dx. \end{aligned} \quad (6.3)$$

The above function has the same form as  $\bar{\Pi}_b(Q)$ , given by Expression (6.1), and, therefore, its unique maximizer  $\bar{Q}_c^*$  satisfies

$$F(\bar{Q}_c^*) = \frac{r + b - p}{r + b - v}. \quad (6.4)$$

**LEMMA 3** The buyer's optimal order quantity, in the centralized model without truck costs, is at least as large as that in the corresponding decentralized model. That is  $\bar{Q}_c^* \geq \bar{Q}_d^*$ .

**Proof:** Follows from Equations (6.2) and (6.4) and the fact that  $p < c$ .

Recall that in both Model I and Model II, the vendor has a generalized replenishment cost structure. Therefore, in both models, the vendor's profits amount to  $\Pi_v(Q) = \bar{\Pi}_v(Q) - \left\lceil \frac{Q}{P_v} \right\rceil R_v$  which in turn leads to

$$\Pi_v(Q) = (c - p)Q - K_v - \lceil Q/P_v \rceil R_v. \quad (6.5)$$

The buyer's subproblems in Model I and Model II are different. Since truck capacity and costs are ignored for the buyer in Model I, the buyer's subproblem in this model is to maximize Expression (6.1). In Model II, however, the buyer wishes to maximize

$$\Pi_b(Q) = \bar{\Pi}_b(Q) - \left\lceil \frac{Q}{P_b} \right\rceil R_b.$$

Under these assumptions, the problem is to decide on the replenishment quantity for the buyer-vendor system under consideration. Next, we discuss how to compute this quantity using the decentralized and centralized modeling approaches. Knowing the properties of the profit expressions in the buyer's and vendor's decentralized subproblems, it is easier to solve the centralized problem where the sum of these two profit functions is maximized. Hence, first we concentrate on the decentralized approach.

## VI.2. Decentralized and Centralized Analysis of the Problem

We begin by presenting some structural properties of the underlying cost and profit functions that are common to both decentralized and centralized models of the problem. For this purpose, let us first consider the following function.

$$h(Q) = g(Q) + \left\lceil \frac{Q}{P} \right\rceil R \quad (6.6)$$

where  $g(Q)$  is a strictly convex function of  $Q$  with a minimizer at  $q$ . The second term of  $h(Q)$  is a stepwise function. Denoting the minimizer of  $h(Q)$  by  $Q^*$ , let us discuss how to find  $Q^*$  over  $Q > 0$ . Hence, the initial feasible region,  $\mathcal{A}$ , for  $Q^*$  is given by

$$\mathcal{A} = \{Q : Q > 0\}.$$

The minimization procedure for  $h(Q)$  will soon be used in optimizing the decentralized and centralized objective functions for Models I and II. Next, we present some properties of  $h(Q)$  that allow us to reduce  $\mathcal{A}$ , simplifying the minimization problem.

**PROPERTY 10** Let  $Q_2 > Q_1 > q$ . Then  $h(Q_2) > h(Q_1)$ . That is,  $h(Q)$  is increasing after  $q$ .

**Proof:** Since  $g(Q)$  is a strictly convex function of  $Q$  and  $q$  is its minimizer,  $g(Q)$  is increasing  $\forall Q > q$ .  $\lceil Q/P \rceil R$  is a nondecreasing function  $\forall Q > q$ . The sum of an increasing and a nondecreasing function is increasing. Therefore,  $h(Q)$  is increasing  $\forall Q > q$ . ■

Property 10 reduces  $\mathcal{A}$  to  $\mathcal{A}^1 = \{Q : 0 < Q \leq q\}$ .

**PROPERTY 11** Let  $Q_1$  and  $Q_2$  be such that  $(k-1)P < Q_1 < Q_2 \leq kP \leq q$  where  $k \geq 1$  or  $(l-1)P < Q_1 < Q_2 \leq q$  where  $l = \lceil q/P \rceil$ . Then  $h(Q_1) > h(Q_2)$ . In other words, for  $Q \leq q$ ,  $h(Q)$  is piece-wise decreasing.

**Proof:** Since  $g(Q)$  is strictly convex with a unique minimizer  $q$ ,  $g(Q_1) > g(Q_2)$ ,  $\forall Q_1, Q_2$  s.t.  $Q_1 < Q_2 \leq q$ . If  $(k-1)P < Q_1 < Q_2 \leq kP$ , we have  $\lceil Q_1/P \rceil = \lceil Q_2/P \rceil = k$ . Therefore,  $g(Q_1) + \lceil Q_1/P \rceil R > g(Q_2) + \lceil Q_2/P \rceil R$ , and hence  $h(Q_1) > h(Q_2)$ . If  $(l-1)P < Q_1 < Q_2 \leq q$  then  $\lceil Q_1/P \rceil = \lceil Q_2/P \rceil = l$ . Therefore,  $g(Q_1) + \lceil Q_1/P \rceil R > g(Q_2) + \lceil Q_2/P \rceil R$ . It follows that  $h(Q_1) > h(Q_2)$ . ■

Property 11 with Property 10 reduces the set within which we should look for the minimizer of  $h(Q)$  to integer multiples of  $P$  that are less than  $q$  and  $q$ . Therefore,  $Q^* \in \mathcal{A}^2 = \{q, Q \text{ s.t. } Q = kP < q \text{ where } k \in \mathbb{Z}^+\}$ . The next property reduces this set further.

**PROPERTY 12** Let us define

$$\mathcal{F} = \{k \in \mathbb{Z}^+ : g(kP) - g((k+1)P) < R, (k+1)P \leq q\}.$$

If  $\mathcal{F} \neq \emptyset$ , let  $i = \min\{k \text{ s.t. } k \in \mathcal{F}\}$ . If  $\mathcal{F} = \emptyset$ , let  $i = 0$ . It follows that, if  $i \neq 0$ , then  $h((j+1)P) > h(jP) \forall j \text{ s.t. } j \geq i \text{ and } (j+1)P \leq q$ .

**Proof:** If  $i \neq 0$ , then  $g(iP) - g((i+1)P) < R$ . Since  $g(Q)$  is a strictly convex function,  $g(jP) - g((j+1)P) < R \forall j \geq i \text{ s.t. } (j+1)P \leq q$ . Since  $h(jP) - h((j+1)P) = g(jP) -$

$g((j+1)P) - R$  and  $g(jP) - g((j+1)P) < R$ , it follows that  $h(jP) - h((j+1)P) < 0$ .

■

Property 12 implies that if  $i \neq 0$ , we do not need to consider integer multiples of  $P$  that are greater than  $iP$ . We also eliminate integer multiples  $P$  that are less than  $iP$ . This is because, if  $g(kP) - g((k+1)P) < R$ , then  $g(kP) + kR < g((k+1)P) + (k+1)R$  and visa versa. This last expression (i.e.,  $g(kP) + kR < g((k+1)P) + (k+1)R$ ) is equivalent to  $h(kP) < h((k+1)P)$ . By definition, if  $i \neq 0$ , then  $i$  is the first integer  $k$  s.t.  $g(kP) - g((k+1)P) < R$  which in turn implies that  $i$  is the first integer  $k$  s.t.  $h(kP) < h((k+1)P)$ . Hence,  $h(iP) < h(jP) \forall j < i$ .

As a result, Properties 10, 11 and 12 imply that

$$Q^* \in \mathcal{A}^3 = \begin{cases} \{q, iP\} & \text{if } i \neq 0, \\ \mathcal{A}^2 & \text{if } i = 0. \end{cases} \quad (6.7)$$

**PROPERTY 13** Let  $l = \lceil q/P \rceil$ . If  $i = 0$ , then either  $(l-1)P$  or  $q$  is optimal.

**Proof:** If  $i = 0$ , then  $g(kP) - g((k+1)P) > R \forall k$  s.t.  $(k+1)P \leq q$ . Therefore,  $h((l-1)P) < h(jP) \forall j < (l-1)$ . This implies that we can eliminate all integer multiples of  $P$  that are less than  $(l-1)P$  from set  $\mathcal{A}^2$  in Expression (6.7). Therefore, either  $(l-1)P$  or  $q$  is optimal. ■

Note that if  $g((l-1)P) - g(q) > R$ , then  $g((l-1)P) + (l-1)R > g(q) + lR$  which leads to  $h((l-1)P) > h(q)$  and hence  $q$  is optimal. If  $g((l-1)P) - g(q) < R$ , then  $g((l-1)P) + (l-1)R < g(q) + lR$  which leads to  $h((l-1)P) < h(q)$ . In this case,  $(l-1)P$  is optimal.

As a result of Property 13,  $\mathcal{A}^3$  reduces to

$$\mathcal{A}^4 = \begin{cases} \{q, iP\} & \text{if } i \neq 0, \\ \{q, (\lceil \frac{q}{P} \rceil - 1)P\} & \text{if } i = 0. \end{cases}$$



**PROPERTY 14** If  $i \neq 0$ , then  $h(q) > h(iP)$  and hence  $iP$  is the minimizer of  $h(Q)$ .

**Proof:** If  $i \neq 0$ , there exists at least one integer  $k$  s.t.  $iP < kP \leq q$ . Suppose that  $h(q) \leq h(iP)$ . Letting  $l = \lceil q/P \rceil$ , this is equivalent to  $g(q) + lR \leq g(iP) + iR$  and hence  $g(iP) - g(q) \geq (l-i)R$ . However, we know from Property 12 that if  $i \neq 0$  then  $g(iP) - g((i+1)P) < R$ ,  $g((i+1)P) - g((i+2)P) < R$ ,  $\dots$ ,  $g((l-1)P) - g(q) < R$ . Therefore,  $g(iP) - g(q) < (l-i)R$ . This contradicts  $g(iP) - g(q) \geq (l-i)R$ . Therefore, if  $i \neq 0$ , then  $h(q) > h(iP)$ . ■

With this final property

$$Q^* \in \mathcal{A}^5 = \begin{cases} \{iP\} & \text{if } i \neq 0, \\ \{q, (\lceil \frac{q}{P} \rceil - 1)P\} & \text{if } i = 0. \end{cases}$$

**COROLLARY 5** The minimizer of  $h(Q)$ ,  $Q^*$ , can take the following values.

$$Q^* = \begin{cases} iP & \text{if } \mathcal{F} \neq \emptyset, \\ \arg \min\{h(q), h((\lceil \frac{q}{P} \rceil - 1)P)\} & \text{if } \mathcal{F} = \emptyset. \end{cases} \quad (6.8)$$

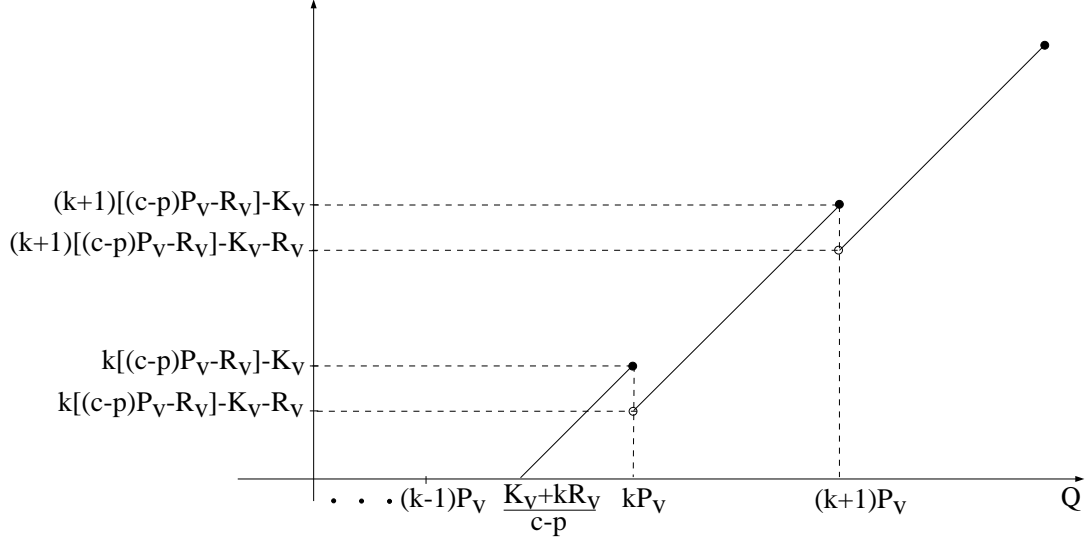
where  $\mathcal{F} = \{k \in Z^+ : g(kP) - g((k+1)P) < R, (k+1)P \leq q\}$  and  $i = \min\{k \text{ s.t. } k \in \mathcal{F}\}$  when  $\mathcal{F} \neq \emptyset$ .

**Proof:** The proof follows from Properties 10–14. ■

### VI.2.1. Vendor's Subproblem

Recall Equation (6.5) which gives an expression of the vendor's profit function  $\Pi_v(Q)$ . Figure 13 provides an illustration of  $\Pi_v(Q)$  based on the following properties of this function.

**PROPERTY 15**  $\Pi_v(Q_2) > \Pi_v(Q_1) \forall Q_1$  and  $Q_2$  s.t.  $(k-1)P_v < Q_1 < Q_2 \leq kP_v$  and  $k \in Z^+$ .

Figure 13 Illustration of  $\Pi_v(Q)$  when  $(c - p)P_v > R_v$ 

**Proof:** For  $(k - 1)P_v < Q \leq kP_v$ , we have

$$\Pi_v(Q) = (c - p)Q - K_v - kR_v,$$

$$d\Pi_v(Q)/dQ > 0$$

Therefore, the function is piecewise increasing over this region, and this completes the proof. ■

**PROPOSITION 8** If  $(c - p)P_v \leq R_v$ , then  $\Pi_v(Q) < 0, \forall Q > 0$ , i.e., the vendor is at loss for any  $Q$  (see Figure 14). If  $(c - p)P_v > R_v$ ,  $\exists Q > 0$  s.t.  $\Pi_v(Q) > 0$ .

**Proof:** It follows from Property 15 that  $\Pi_v(kP_v) > \Pi_v(Q) \forall Q$  s.t.  $(k - 1)P_v < Q \leq kP_v$  where  $k \in \mathbb{Z}^+$ . From Expression (6.5) we also have  $\Pi_v(kP_v) = k[(c - p)P_v - R_v] - K_v$ . If  $(c - p)P_v \leq R_v$  then  $\Pi_v(kP_v) < 0 \forall k \in \mathbb{Z}^+$ . Since  $\Pi_v(kP_v) > \Pi_v(Q) \forall Q$  s.t.  $(k - 1)P_v < Q \leq kP_v$  and  $k \in \mathbb{Z}^+$ , it turns out that  $\Pi_v(Q) < 0 \forall Q > 0$ .

For the second part of the proposition, consider  $Q = kP_v$  where

$$k = \lceil K_v / ((c-p)P_v - R_v) \rceil.$$

Since  $(c-p)P_v > R_v$ , we have  $k \in Z^+$ . Therefore,

$$\begin{aligned} \Pi_v(kP_v) &= \lceil K_v / ((c-p)P_v - R_v) \rceil [(c-p)P_v - R_v] - K_v \\ &\geq (K_v / ((c-p)P_v - R_v)) [(c-p)P_v - R_v] - K_v = 0. \end{aligned}$$

■

**PROPERTY 16** If  $(c-p)P_v > R_v$ , then  $\Pi_v((k+1)P_v) > \Pi_v(kP_v) \forall k \in Z^+$ . That is, if  $(c-p)P_v > R_v$ , the vendor's profits at integer multiples of  $P_v$  are increasing.

**Proof:** From Expression (6.5), we have  $\Pi_v((k+1)P_v) - \Pi_v(kP_v) = (c-p)(k+1)P_v - K_v - (k+1)R_v - (c-p)kP_v + K_v + kR_v = (c-p)P_v - R_v$ . Since  $(c-p)P_v > R_v$ , it follows that  $(c-p)P_v - R_v > 0$  and, hence,  $\Pi_v((k+1)P_v) - \Pi_v(kP_v) > 0$ . ■

**PROPERTY 17** If  $(c-p)P_v > R_v$ , then  $\Pi_v(kP_v) = \Pi_v(kP_v + \frac{R_v}{c-p}) > \Pi_v(Q) \forall Q$  s.t.  $kP_v < Q < kP_v + \frac{R_v}{c-p}$ ,  $k \in Z^+$ .

**Proof:** Since  $Q < kP_v + \frac{R_v}{c-p}$ , we have  $(c-p)Q < (c-p)kP_v + R_v$ . This in turn implies that  $(c-p)Q - (k+1)R_v < (c-p)kP_v - kR_v$ . Subtracting  $K_v$  from both sides of this inequality leads to  $(c-p)Q - K_v - (k+1)R_v < (c-p)kP_v - K_v - kR_v$ . By assumption,  $kP_v < Q < kP_v + \frac{R_v}{c-p}$  and  $(c-p)P_v > R_v$  (i.e.  $\frac{R_v}{c-p} < P_v$ ), so that we also have  $\lceil \frac{Q}{P_v} \rceil = k+1$ . Therefore,  $(c-p)Q - K_v - \lceil \frac{Q}{P_v} \rceil R_v < (c-p)kP_v - K_v - kR_v$  which implies that

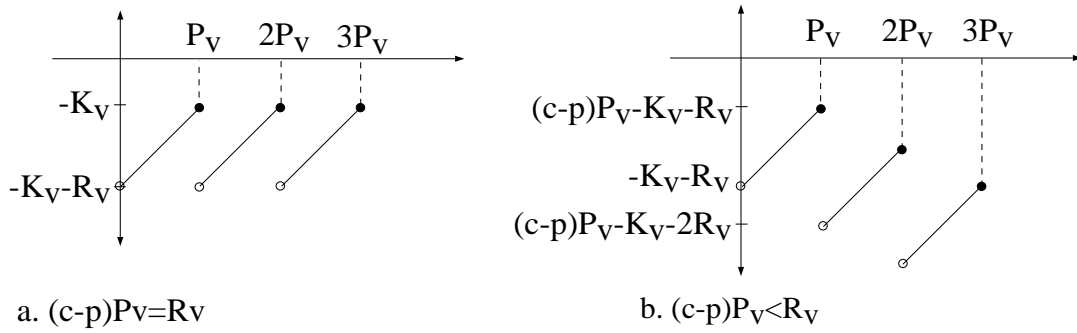
$$\Pi_v(kP_v) > \Pi_v(Q) \quad \forall Q \text{ s.t. } kP_v < Q < kP_v + \frac{R_v}{c-p}.$$

Note that if  $Q = kP_v + \frac{R_v}{c-p}$ , then

$$\Pi_v(Q) = (c-p)kP_v + R_v - K_v - (k+1)R_v = (c-p)kP_v - K_v - kR_v = \Pi_v(kP_v).$$

■

Figure 14 Different illustrations of  $\Pi_v(Q)$



### VI.2.2. Decentralized and Centralized Decision Problems for Model I

As described in Section VI.1, the buyer's decentralized decision problem for Model I is to find the value of  $Q$  that maximizes  $\bar{\Pi}_b(Q)$  given by Expression (6.1). Let  $Q_{d,1}^*$  denote the optimal solution of this problem. Obviously  $Q_{d,1}^* = \bar{Q}_d^*$ .

The objective function to be maximized in the centralized model is the sum of the expected vendor profits and expected buyer profits which in turn is given by

$$\Pi_c^I(Q) = \Pi_v(Q) + \bar{\Pi}_b(Q). \quad (6.9)$$

Noting that  $\Pi_v(Q) = \bar{\Pi}_v(Q) - \left\lceil \frac{Q}{P_v} \right\rceil R_v$ , this function can be expressed as

$$\Pi_c^I(Q) = \bar{\Pi}_b(Q) + \bar{\Pi}_v(Q) - \left\lceil \frac{Q}{P_v} \right\rceil R_v \quad (6.10)$$

Using the fact that  $\bar{\Pi}_c(Q) = \bar{\Pi}_v(Q) + \bar{\Pi}_b(Q)$ , the above expression reduces to

$$\Pi_c^I(Q) = \bar{\Pi}_c(Q) - \left\lceil \frac{Q}{P_v} \right\rceil R_v \quad (6.11)$$

Recall from Section VI.1 that  $\bar{\Pi}_c(Q)$ , given by Expression (6.3), is the expected system profits of the buyer-vendor system without truck capacity and costs. This is a strictly concave function whose maximizer  $\bar{Q}_c^*$  is given by Expression (6.4). Denoting the optimum level of the buyer's order quantity in the centralized solution of Model I by  $Q_{c,1}^*$ , we have the following property

**PROPERTY 18** The following are true for the objective function values of Model I with and without truck costs:

1.  $\Pi_c^I(Q_{c,1}^*) < \bar{\Pi}_c(\bar{Q}_c^*)$ .
2. If  $\bar{\Pi}_c(\bar{Q}_c^*) < R$  then  $\Pi_c^I(Q_{c,1}^*) < 0$ .

**Proof:**

1. Expression (6.11) implies that  $\Pi_c^I(Q) < \bar{\Pi}_c(Q) \forall Q > 0$ . As a result,

$$\max_{Q>0} \Pi_c^I(Q) = \Pi_c^I(Q_{c,1}^*) < \bar{\Pi}_c(Q_{c,1}^*) \leq \max_{Q>0} \bar{\Pi}_c(Q) = \bar{\Pi}_c(\bar{Q}_c^*).$$

2. Again, using Equation (6.11) we have

$$\max_{Q>0} \Pi_c^I(Q) = \max_{Q>0} \left\{ \bar{\Pi}_c(Q) - \left\lceil \frac{Q}{P} \right\rceil R \right\}$$

which satisfies

$$\max_{Q>0} \left\{ \bar{\Pi}_c(Q) - \left\lceil \frac{Q}{P} \right\rceil R \right\} \leq \max_{Q>0} \bar{\Pi}_c(Q) - \min_{Q>0} \left\lceil \frac{Q}{P} \right\rceil R.$$

Since over  $Q > 0$ ,  $\min \left\lceil \frac{Q}{P} \right\rceil R = R$ , the above inequality leads to  $\Pi_c^I(Q_{c,1}^*) \leq \bar{\Pi}_c(\bar{Q}_c^*) - R$ . Consequently, if  $\bar{\Pi}_c(\bar{Q}_c^*) < R$  then  $\Pi_c^I(Q_{c,1}^*) < 0$ . ■

Note that maximizing  $\Pi_c^I(Q)$ , given by (6.11), is equivalent to minimizing  $-\bar{\Pi}_c(Q) + \left\lceil \frac{Q}{P_v} \right\rceil R_v$ . This sum consists of a strictly convex function (i.e.  $-\bar{\Pi}_c(Q)$ ) and a stepwise cost function as  $h(Q)$  in Expression (6.6). Therefore, the minimizer can be computed using Expression (6.8) by substituting  $g(Q) = -\bar{\Pi}_c(Q)$  and  $q = \bar{Q}_c^*$  so that  $h(Q) = -\Pi_c^I(Q)$ . As a result,

$$Q_{c,1}^* = \begin{cases} iP_v & \text{if } \mathcal{F} \neq \emptyset, \\ \arg \max \left\{ \Pi_c^I(\bar{Q}_c^*), \Pi_c^I \left( \left( \left\lceil \frac{\bar{Q}_c^*}{P_v} \right\rceil - 1 \right) P_v \right) \right\} & \text{if } \mathcal{F} = \emptyset. \end{cases} \quad (6.12)$$

where  $\mathcal{F} = \{k \in Z^+ : -\bar{\Pi}_c(kP_v) + \bar{\Pi}_c((k+1)P_v) < R_v, (k+1)P_v \leq \bar{Q}_c^*\}$  and  $i = \min\{k \text{ s.t. } k \in \mathcal{F}\}$  when  $\mathcal{F} \neq \emptyset$ .

**REMARK 2** It follows from Expression (6.12) that  $Q_{c,1}^* \leq \bar{Q}_c^*$ . That is, the centralized order quantity of the system considering truck capacity and costs for the vendor, is at most as great as that of the system without considering truck capacity and costs.

**THEOREM 4** Let  $\mathcal{F}^o = \{k : -\bar{\Pi}_c(kP_v) + \bar{\Pi}_c((k+1)P_v) < R_v\}$ , then  $\forall k \in \mathcal{F}^o$  we have the following inequality.

$$F((k+1)P_v) > \frac{(r-p+b)P_v - R_v + (r-v+b) \int_{kP_v}^{(k+1)P_v} (x - kP_v) f(x) dx}{(r-v+b)P_v} \quad (6.13)$$

where  $F(\cdot)$  and  $f(\cdot)$  denote the distribution and density functions of demand, respectively.

**Proof:**

Recalling Expression (6.3), we can write

$$\begin{aligned} -\bar{\Pi}_c(kP_v) + \bar{\Pi}_c((k+1)P_v) &= (v-p)P_v + (r-v+b) \int_{kP_v}^{(k+1)P_v} (x - kP_v) f(x) dx + \\ &\quad + (r-v+b)P_v [1 - F((k+1)P_v)]. \end{aligned}$$

If  $k \in \mathcal{F}^o$ , then  $-\bar{\Pi}_c(kP_v) + \bar{\Pi}_c((k+1)P_v) < R_v$ . Combining the above two expressions and rearranging the terms leads to (6.13).  $\blacksquare$

Note that in Inequality (6.13), the expression in the numerator can be considered the system's cost associated with not ordering another full truck load in addition to  $k$  full trucks. This is similar to the underage cost of each unit demand that cannot be met. Similarly,  $(p-v)P_v + R_v - (r-v+b) \int_{kP_v}^{(k+1)P_v} (x-kP_v)f(x)dx$  can be interpreted as the cost associated with ordering an additional full truck load in excess of  $k$  full trucks. Hence, the denominator of Inequality (6.13) can be interpreted as the sum of overage and underage system costs associated with a truck load in addition to  $k$  trucks.

Based on the vendor's cost parameters and the properties described in Section VI.2.1, there are some special cases where Expression (6.12) can be simplified further. Theorem 5 and Propositions 9 and 10 discuss such cases.

**THEOREM 5** Suppose  $(c-p)P_v \geq R_v$ .

- If  $Q_{d,1}^* \neq \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v$  (i.e.,  $Q_{d,1}^*$  is not a full truck load), then  $Q_{c,1}^* \geq \left( \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1 \right) P_v$ .
- If  $Q_{d,1}^* = \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v$ , then  $Q_{c,1}^* \geq Q_{d,1}^*$ .

**Proof:** Using Expression (6.10) and the fact that  $\bar{\Pi}_v(Q) = (c-p)Q - K_v$ , we have

$$\Pi_c^I(Q) = \bar{\Pi}_b(Q) + (c-p)Q - K_v - \lceil Q/P_v \rceil R_v \quad (6.14)$$

Recall that  $\bar{\Pi}_b(Q)$  is a strictly concave function with a maximizer at  $\bar{Q}_d^*$ . Since truck capacity and costs are ignored for the buyer in Model I,  $Q_{d,1}^* = \bar{Q}_d^*$ .

- For the first part of the proof, we will show that  $\forall m \in Z^+$  and  $m < j-1$ ,  $\Pi_c^I(mP_v) < \Pi_c^I((j-1)P_v)$  where  $j = \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil$ . Let's consider  $\Pi_c^I((j-1)P_v) - \Pi_c^I(mP_v) = \bar{\Pi}_b((j-1)P_v) - \bar{\Pi}_b(mP_v) + [(c-p)P_v - R_v](j-1-m)$ . Since

$Q_{d,1}^*$  is the maximizer of  $\bar{\Pi}_b(Q)$  and  $\bar{\Pi}_b(Q)$  is a strictly concave function of  $Q$ ,  $\bar{\Pi}_b((j-1)P_v) - \bar{\Pi}_b(mP_v) > 0$ . When  $(c-p)P_v > R_v$ , it is also true that  $[(c-p)P_v - R_v](j-1-m) > 0$ . Therefore,  $\Pi_c^I((j-1)P_v) - \Pi_c^I(mP_v) > 0$ . Hence,  $Q_{c,1}^* \geq (j-1)P_v$ .

- For the second part of the proof, we will show that  $\forall m \in Z^+$  and  $m < j$ ,  $\Pi_c^I(mP_v) < \Pi_c^I(jP_v)$ . For this purpose, let's consider the difference  $\Pi_c^I(jP_v) - \Pi_c^I(mP_v) = \bar{\Pi}_b(jP_v) - \bar{\Pi}_b(mP_v) + [(c-p)P_v - R_v](j-m)$ . Using the concavity of  $\bar{\Pi}_b(Q)$  and the fact that  $mP_v < jP_v = Q_{d,1}^*$ , we have that  $\bar{\Pi}_b(jP_v) - \bar{\Pi}_b(mP_v) > 0$ . When  $(c-p)P_v > R_v$ ,  $[(c-p)P_v - R_v](j-m) > 0$ . Therefore,  $\Pi_c^I(jP_v) - \Pi_c^I(mP_v) > 0$  and, hence, if  $Q_{d,1}^* = \left\lceil \frac{Q_d^*}{P_v} \right\rceil P_v$ , then  $Q_{c,1}^* \geq Q_{d,1}^*$ . ■

The above theorem simplifies the computation of  $Q_{c,1}^*$  given by (6.12) in the following way. When  $(c-p)P_v \geq R_v$ , we do not need to consider certain values for  $i$ . That is, we compute  $Q_{d,1}^*$ , and if  $Q_{d,1}^* \neq \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v$ , then we construct  $\mathcal{F}$  by checking the conditions  $-\bar{\Pi}_c(kP_v) + \bar{\Pi}_c((k+1)P_v) < R_v$  and  $(k+1)P_v \leq \bar{Q}_c^*$  for  $k \geq \left( \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1 \right)$ . On the other hand, if  $Q_{d,1}^* = \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v$ , then we do the same for  $k \geq \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil$ .

**COROLLARY 6** If  $(c-p)P_v \geq R_v$ , the only possible value of  $Q_{c,1}^*$  that is less than  $Q_{d,1}^*$  is  $\left( \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1 \right) P_v$ .

**Proof:** Proof follows from Expression (6.12) and Theorem 5. ■

**PROPOSITION 9** When  $(c-p)P_v > R_v$  and  $Q_{d,1}^* \geq \left( \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1 \right) P_v + \frac{R_v}{c-p}$ , then  $Q_{c,1}^* \geq Q_{d,1}^*$ .

**Proof:** As stated in Corollary 6, when  $(c-p)P_v \geq R_v$ , the only possible value of  $Q_{c,1}^*$  that is less than  $Q_{d,1}^*$  is  $\left( \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1 \right) P_v$ . Since  $(c-p)P_v > R_v$ , from Property 17, we



have

$$\begin{aligned} \Pi_v(Q) &\geq \Pi_v \left( \left( \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1 \right) P_v \right) \\ \forall Q \text{ s.t. } \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v &\geq Q > \left( \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1 \right) P_v + \frac{R_v}{c-p}. \\ \text{Therefore, } \Pi_v(Q_{d,1}^*) &\geq \Pi_v \left( \left( \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1 \right) P_v \right). \text{ Since } Q_{d,1}^* \text{ is the maximizer of } \\ \bar{\Pi}_b(Q), \text{ we also have} \end{aligned}$$

$$\bar{\Pi}_b(Q_{d,1}^*) > \bar{\Pi}_b \left( \left( \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1 \right) P_v \right).$$

Hence from Expression (6.9), we conclude that

$$\Pi_c^I(Q_{d,1}^*) > \Pi_c^I \left( \left( \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1 \right) P_v \right).$$

■

**PROPOSITION 10** When  $(c-p)P_v = R_v$ ,

- If  $Q_{d,1}^* \neq \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v$ , then  $\left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v \geq Q_{c,1}^* \geq \left( \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1 \right) P_v$ .
- If  $Q_{d,1}^* = \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v$ , then  $Q_{c,1}^* = Q_{d,1}^*$ .

**Proof:**

- As illustrated in 14.a, if  $(c-p)P_v = R_v$ , then  $\Pi_v(kP_v) \geq \Pi_v(Q)$ ,  $\forall Q$  and  $\forall k \in Z^+$ . Therefore,  $\Pi_v(Q)$  is maximized at both  $Q = \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v$  and  $Q = \left( \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1 \right) P_v$ . Hence,  $\forall Q > 0$  we have

$$\Pi_v \left( \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v \right) \geq \Pi_v(Q) \quad (6.15)$$

and

$$\Pi_v \left( \left( \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1 \right) P_v \right) \geq \Pi_v(Q). \quad (6.16)$$

Let's first consider  $Q > \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v$ . Since  $\bar{\Pi}_b(Q)$  is a strictly concave function with a maximizer at  $Q_{d,1}^*$  and  $Q_{d,1}^* \leq \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v < Q$ , we have

$$\bar{\Pi}_b\left(\left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v\right) > \bar{\Pi}_b(Q). \quad (6.17)$$

Using Expressions (6.9), (6.15) and (6.17), we conclude that

$$\Pi_c^I\left(\left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v\right) > \Pi_c^I(Q), \quad \forall Q > \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v.$$

With a similar argument, it can also be shown that

$$\Pi_c^I\left(\left(\left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1\right) P_v\right) > \Pi_c^I(Q), \quad \forall Q < \left(\left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1\right) P_v.$$

Therefore,  $\left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v \geq Q_{c,1}^* \geq \left(\left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil - 1\right) P_v$ .

- If  $Q_{d,1}^* = \left\lceil \frac{Q_{d,1}^*}{P_v} \right\rceil P_v$ , since both  $\bar{\Pi}_b(Q)$  and  $\Pi_v(Q)$  are maximized at  $Q = Q_{d,1}^*$ ,  $\Pi_c^I(Q)$  is also maximized at this value. Therefore,  $Q_{c,1}^* = Q_{d,1}^*$ .

■

For some special demand densities, the solution to the centralized model can be further reduced. We next analyze the solution procedure for exponential and uniform demand distributions.

### Exponential Demand Distribution:

Assume that  $f(x) = \lambda e^{-\lambda x}$  if  $x \geq 0$ . The buyer's expected profit function can be obtained using Expression (6.1) as

$$\bar{\Pi}_b(Q) = (-c + v)Q - K_b + \frac{r - v}{\lambda} - (r - v + b)\frac{e^{-\lambda Q}}{\lambda} \quad (6.18)$$

Expected system profit is then given by

$$\Pi_c^I(Q) = (v - p)Q - K_b - K_v + \frac{r - v}{\lambda} - (r - v + b) \frac{e^{-\lambda Q}}{\lambda} - \left\lceil \frac{Q}{P_v} \right\rceil R_v \quad (6.19)$$

Recall that  $Q_{d,1}^* = \bar{Q}_d^*$ ; therefore,  $Q_{d,1}^*$  can be derived using Expression (6.2). This leads to

$$Q_{d,1}^* = -\frac{1}{\lambda} \ln \left( \frac{c - v}{r + b - v} \right) \quad (6.20)$$

The expression for  $\bar{Q}_c^*$  can be evaluated from Equality (6.4) and is given by

$$\bar{Q}_c^* = -\frac{1}{\lambda} \ln \left( \frac{p - v}{r + b - v} \right) \quad (6.21)$$

**PROPOSITION 11** Letting  $\hat{n} = \max \left\{ 1, \left\lceil -\frac{1}{\lambda P} \ln \left( \frac{\lambda(R_v + (p-v)P_v)}{(1-e^{-\lambda P})(r-v+b)} \right) \right\rceil \right\}$ ,

$$Q_{c,1}^* = \begin{cases} \hat{n}P_v & \text{if } (\hat{n} + 1)P_v \leq \bar{Q}_c^*, \\ \arg \max \left\{ \Pi_c^I(\bar{Q}_c^*), \Pi_c^I \left( \left( \left\lceil \frac{\bar{Q}_c^*}{P_v} \right\rceil - 1 \right) P_v \right) \right\} & \text{o.w.} \end{cases} \quad (6.22)$$

**Proof:** Using Theorem 4, it can be shown that  $k \in \mathcal{F}^o$  where  $\mathcal{F}^o = \{k : -\bar{\Pi}_c(kP_v) + \bar{\Pi}_c((k+1)P_v) < R_v\}$  satisfies the following inequality

$$k > -\frac{1}{\lambda P_v} \ln \left( \frac{\lambda(R_v + (p-v)P_v)}{(1-e^{-\lambda P_v})(r-v+b)} \right)$$

If the right hand side of the above inequality is less than zero, then it is satisfied by all positive integer values of  $k$ . However, we need to find the smallest positive such integer. In this case,  $\hat{n} = 1$ . When the right hand side is greater than zero, then  $\hat{n}$  is given by the smallest integer that is greater than this value. Additionally, if  $\hat{n} + 1$  is less than or equal to  $\bar{Q}_c^*$ , then  $\hat{n}$  satisfies  $Q_{c,1}^* = \hat{n}P_v$ . ■

### Uniform Demand Distribution:

Assume that demand is uniformly distributed over the interval  $(d_1, d_2)$ . That is,  $f(x) = 1/(d_2 - d_1)$  for  $d_1 < x < d_2$ . Again  $Q_{d,1}^*$  and  $\bar{Q}_c^*$  can be derived using Expressions (6.2) and (6.4) as

$$Q_{d,1}^* = \frac{r + b - c}{r + b - v}(d_2 - d_1) + d_1,$$

and

$$\bar{Q}_c^* = \frac{r + b - p}{r + b - v}(d_2 - d_1) + d_1. \quad (6.23)$$

Using Expressions (6.1) and (6.9), it can easily be shown that

$$\bar{\Pi}_b(Q) = \begin{cases} (r + b - c)Q - b\frac{d_1+d_2}{2} - K_b, & \text{if } Q \leq d_1, \\ (-c + v)Q - K_b + \frac{(r-v)(d_1+d_2)}{2} - \frac{(r-v+b)}{2(d_2-d_1)}(d_2 - Q)^2, & \text{if } d_1 < Q < d_2 \\ (-c + v)Q - K_b + (r - v)\frac{d_1+d_2}{2}, & \text{if } Q \geq d_2 \end{cases} \quad (6.24)$$

and

$$\Pi_c^I(Q) = \begin{cases} (r + b - p)Q - b\frac{d_1+d_2}{2} - \left\lceil \frac{Q}{P_v} \right\rceil R_v - K_b - K_v, & \text{if } Q \leq d_1, \\ (v - p)Q + \frac{(r-v)(d_1+d_2)}{2} - \frac{(r-v+b)}{2(d_2-d_1)}(d_2 - Q)^2 - \left\lceil \frac{Q}{P_v} \right\rceil R_v \\ \quad - K_b - K_v, & \text{if } d_1 < Q < d_2 \\ (v - p)Q + (r - v)\frac{d_1+d_2}{2} - \left\lceil \frac{Q}{P_v} \right\rceil R_v - K_b - K_v, & \text{if } Q \geq d_2 \end{cases} \quad (6.25)$$

For the case of uniform demand, Expression (6.12) can be simplified as we discuss below.

**PROPOSITION 12** Let  $\mathcal{F}^o = \{k : -\bar{\Pi}_c(kP_v) + \bar{\Pi}_c((k+1)P_v) < R_v\}$ . If demand is uniformly distributed over  $(d_1, d_2)$ , then the following are true.

1. If  $kP_v < (k+1)P_v \leq d_1$ ,  $k \notin \mathcal{F}^o$  unless  $(r - p + b)P_v < R_v$ .

2. If  $kP_v \leq d_1 < (k+1)P_v \leq \bar{Q}_c^* \leq d_2$ ,  $k \notin \mathcal{F}^o$  unless

$$\frac{AP_v^2 k^2}{2} + A(P_v^2 - P_v d_1)k + B > 0 \quad (6.26)$$

where  $A = \frac{r-v+b}{d_2-d_1}$  and  $B = (p-v)P_v + R_v + \frac{r-v+b}{d_2-d_1} \left[ \frac{P_v^2}{2} - P_v d_2 + \frac{d_1^2}{2} \right]$ .

3. If  $d_1 < kP_v < (k+1)P_v \leq \bar{Q}_c^* \leq d_2$ ,  $k \notin \mathcal{F}^o$  unless

$$k > \frac{2d_2 - P_v}{2P_v} - \frac{(d_2 - d_1)[R_v + (p-v)P_v]}{(r-v+b)P_v^2}. \quad (6.27)$$

**Proof:** Proof follows from Theorem 4. ■

**PROPOSITION 13** The following algorithm can be used to maximize  $\Pi_c^I(Q)$  when demand is uniformly distributed over the interval  $(d_1, d_2)$ .

1. Compute  $\bar{Q}_c^*$  using (6.23).
2. If  $2P_v > \bar{Q}_c^*$ , go to Step 6; otherwise go to Step 3.
3. If  $(r-p+b)P_v < R_v$  continue; otherwise go to Step 5.
4. Let  $\hat{n} = \max \left\{ 1, \left\lceil \frac{2d_2 - P_v}{2P_v} - \frac{(d_2 - d_1)[R_v + (p-v)P_v]}{(r-v+b)P_v^2} \right\rceil \right\}$ . If  $d_1 < P_v < 2d_1$  continue; otherwise set  $Q_{c,1}^* = P_v$  and stop.
  - (a) If  $(\hat{n} + 1) \leq \bar{Q}_c^*$ , set  $Q_{c,1}^* = \hat{n}P_v$  and stop.
  - (b) If  $(\hat{n} + 1) > \bar{Q}_c^*$ , go to Step 6.
5. If  $P_v \leq d_1$ , go to Step 5.a; otherwise go to Step 5.b.
  - (a) Let  $m = \left\lfloor \frac{d_1}{P_v} \right\rfloor$ . Note that  $m \geq 1$ . If  $(m+1)P_v > \bar{Q}_c^*$ , go to Step 6. Otherwise, if  $m$  satisfies Inequality (6.26), set  $Q_{c,1}^* = mP_v$  and stop. Otherwise continue.
  - (b) Let  $m = \left\lfloor \frac{d_1}{P_v} \right\rfloor$  and define  $\hat{n} = \max \left\{ m, \left\lceil \frac{2d_2 - P_v}{2P_v} - \frac{(d_2 - d_1)[R_v + (p-v)P_v]}{(r-v+b)P_v^2} \right\rceil \right\}$ . If  $(\hat{n} + 1)P_v > \bar{Q}_c^*$ , go to Step 6. Otherwise set  $Q_{c,1}^* = mP_v$  and stop.

6. Set  $Q_{c,1}^*$  either to  $\bar{Q}_c^*$  or  $\left(\left\lceil \frac{\bar{Q}_c^*}{P_v} \right\rceil - 1\right) P_v$  whichever gives the maximum profit.

**Proof:** Step 2 follows directly from Expression (6.8). Steps 3 and 4 are executed only if  $(r - p + b)P_v < R_v$  and Step 5 is executed only if  $(r - p + b)P_v \geq R_v$ . Therefore, we will analyze these two cases separately. Note that at this point we should have  $2P_v \leq \bar{Q}_c^*$ .

Case 1,  $(r - p + b)P_v < R_v$ : Here we have the following subcases. Note that, since  $\bar{Q}_c^* \leq d_2$  and  $2P_v \leq \bar{Q}_c^*$  we also have  $2P_v \leq d_2$ .

1.  $2P_v \leq d_1$ : From Item 1) of Proposition 12,  $k = 1$  satisfies the condition that  $-\bar{\Pi}_c(kP_v) + \bar{\Pi}_c((k+1)P_v) < R_v$ . Since  $d_1 < \bar{Q}_c^*$ , we have  $2P_v < \bar{Q}_c^*$ . Therefore,  $k = 1$  is the minimum element of the set  $\mathcal{F}$  in Expression (6.12). Hence,  $i = 1$  which implies  $Q_{c,1}^* = P_v$ .
2.  $P_v \leq d_1 < 2P_v$ : If this condition is satisfied, then  $Q_{c,1}^* = P_v$ . In proving this, we will use Item 2) of Proposition 12. The left side of the Inequality (6.26) is a convex function  $k$  because  $A > 0$ . Its minimum value is  $-\frac{A^2(P_v^2 - P_v d_1)^2}{2AP_v^2} + B$ . If  $(r + b - p)P_v < R_v$ , then this value is greater than zero. Therefore, the function is positive everywhere. This implies that Inequality (6.26) is satisfied by all values of  $k$ . However, we need the minimum integer  $k$  and the natural candidate is  $k = 1$ . Since  $k = 1$  also satisfies the condition that  $(k+1)P_v \leq \bar{Q}_c^*$ , we have  $i = 1$  and  $Q_{c,1}^* = P_v$ .
3.  $P_v \geq 2d_1$ : If this condition is satisfied, then  $Q_{c,1}^* = P_v$ . In proving this, we will use Item 3) of Proposition 12. If  $P_v \geq 2d_1$  in addition to  $(r+b-p)P_v < R_v$ , it can be shown that the right hand side of Inequality (6.27) is negative. Therefore, it's satisfied by all positive real numbers  $k$ . However, we need the smallest integer value and again the natural candidate is  $k = 1$ . Since  $k = 1$  also satisfies the

condition that  $(k+1)P_v \leq \bar{Q}_c^*$ , we have  $i = 1$  and  $Q_{c,1}^* = P_v$ .

4.  $d_1 < P_v < 2d_1$ : Since  $P_v > d_1$ , we will again use Item 3) of Proposition 12.

However, now we cannot conclude anything about whether the right hand side of Inequality (6.27) is negative or positive. If it's negative, then  $\hat{n} = 1$  and  $Q_{c,1}^* = P_v$ . If it's positive, then we should also check if  $(\hat{n} + 1) \leq \bar{Q}_c^*$ . If this condition is also satisfied, then  $i = \hat{n}$ . Otherwise Step 6 is executed and this follows from Expression (6.8).

Case 2,  $(r - p + b)P_v \geq R_v$ : We will again use Proposition 12. Since  $(r - p + b)P_v \geq R_v$ , Item 1) does not hold. Therefore, the candidate values of  $k$  for which  $-\bar{\Pi}_c(kP_v) + \bar{\Pi}_c((k+1)P_v) < R_v$  is satisfied are implied by Item 2) or Item 3) of Proposition 12. We have two cases:  $P_v \leq d_1$  and  $P_v > d_1$ .

If  $P_v \leq d_1$ , the only possible value that is implied by Item 2 is  $m = \left\lfloor \frac{d_1}{P_v} \right\rfloor$ . If  $(m+1)P_v > \bar{Q}_c^*$ , then we have  $(k+1)P_v > \bar{Q}_c^* \forall k > m$ . Therefore, we do not need to check Item 3). If  $(m+1)P_v \leq \bar{Q}_c^*$  and  $m$  satisfies Inequality (6.26), then  $i = m$ . On the other hand, if  $m$  does not satisfy Inequality (6.26), we still need to check integer values that are greater than  $m$ . Note that  $-\bar{\Pi}_c(kP_v) + \bar{\Pi}_c((k+1)P_v) < R_v$  can be satisfied by these values only if Item 3) of Proposition 12 holds. If  $P_v > d_1$ , we again use the same condition. But we need the smallest integer  $k$  that is implied by Item 3). This is given by  $\hat{n} = \max \left\{ m, \left\lceil \frac{2d_2 - P_v}{2P_v} - \frac{(d_2 - d_1)[R_v + (p-v)P_v]}{(r-v+b)P_v^2} \right\rceil \right\}$ . If  $(\hat{n} + 1)P_v \leq \bar{Q}_c^*$ , we have  $i = m$  and  $Q_{c,1}^* = mP_v$ . Otherwise, Step 6 follows from Expression (6.8). ■

### VI.2.3. Coordinated Solution for Model I

In this section, we propose two coordination mechanisms by which the buyer orders the centralized order quantity while achieving the expected profits in his/her decentralized solution. Propositions 15 and 16 describe the structure of the first co-

ordination mechanism. Proposition 14 is useful in proving why such coordination mechanisms work. To this end, with a slight change of notation, we use  $\bar{\Pi}_b(Q, c)$  for the expected buyer profit function. This is because the wholesale price  $c$  will be specified by the vendor in such a way that ordering the centralized order quantity does not decrease the buyer's profits relative to his/her decentralized ordering policy. Therefore, we treat  $c$  as a decision variable, and we let  $\bar{Q}_d^*(\cdot)$  represent the optimal decentralized order quantity in Model I for a given value of the wholesale price. In the remainder of the text, we let, as before,  $\bar{Q}_d^*(c) = \bar{Q}_d^*$ .

**PROPOSITION 14** Let  $v < c_l < c^o < c_h$  where  $c_l$  and  $c_h$  represent lower and higher price values in comparison to  $c^o$ . Then  $\bar{\Pi}_b(\bar{Q}_d^*(c_l), c_l) > \bar{\Pi}_b(\bar{Q}_d^*(c^o), c^o)$  and  $\bar{\Pi}_b(\bar{Q}_d^*(c_h), c_h) < \bar{\Pi}_b(\bar{Q}_d^*(c^o), c^o)$ .

**Proof:**  $\bar{\Pi}_b(\bar{Q}_d^*(c_l), c_l) > \bar{\Pi}_b(\bar{Q}_d^*(c^o), c_l)$  because by definition  $\bar{Q}_d^*(c_l)$  maximizes  $\bar{\Pi}_b(Q, c_l)$ . For a fixed value of  $Q$ ,  $\bar{\Pi}_b(Q, c)$  is decreasing in  $c$ . Therefore,  $\bar{\Pi}_b(\bar{Q}_d^*(c^o), c_l) > \bar{\Pi}_b(\bar{Q}_d^*(c^o), c^o)$  and hence

$$\bar{\Pi}_b(\bar{Q}_d^*(c_l), c_l) > \bar{\Pi}_b(\bar{Q}_d^*(c^o), c_l) > \bar{\Pi}_b(\bar{Q}_d^*(c^o), c^o).$$

Similarly,  $\bar{\Pi}_b(\bar{Q}_d^*(c^o), c^o) > \bar{\Pi}_b(\bar{Q}_d^*(c_h), c^o)$ , because by definition  $\bar{Q}_d^*(c^o)$  maximizes  $\bar{\Pi}_b(Q, c^o)$ . Since  $c_h > c^o$ , we also have  $\bar{\Pi}_b(\bar{Q}_d^*(c_h), c^o) > \bar{\Pi}_b(\bar{Q}_d^*(c_h), c_h)$ . It follows that

$$\bar{\Pi}_b(\bar{Q}_d^*(c^o), c^o) > \bar{\Pi}_b(\bar{Q}_d^*(c_h), c^o) > \bar{\Pi}_b(\bar{Q}_d^*(c_h), c_h).$$

■

**PROPOSITION 15** Let  $\Delta_1 = (r + b - v) [F(Q_{c,1}^*) - F(\bar{Q}_d^*)]$  and  $c_1 = c - \Delta_1$ . If  $Q_{c,1}^* > \bar{Q}_d^*$ , under a unit discount of  $\Delta_1$  offered by the vendor to the buyer and a franchise fee of  $\bar{\Pi}_b(Q_{c,1}^*, c_1) - \bar{\Pi}_b(\bar{Q}_d^*, c)$  paid by the buyer to the vendor, the buyer stays in a “no worse” situation by ordering  $Q_{c,1}^*$  units.



**Proof:** By assumption, demand (i.e.  $X$ ) is a continuous nonnegative random variable. We define  $\alpha_1 = \inf\{x : f(x) > 0\}$  and  $\alpha_2 = \sup\{x : f(x) > 0\}$ . Therefore,  $f(x) > 0$  where  $\alpha_1 < x < \alpha_2$ . Under the classical Newsboy assumptions, it is true that  $\alpha_1 < \bar{Q}_d^* < \alpha_2$  and  $\alpha_1 < \bar{Q}_c^* < \alpha_2$ . From Remark 2, we know that  $Q_{c,1}^* \leq \bar{Q}_c^*$ . Therefore,  $\bar{Q}_d^* < Q_{c,1}^* \leq \bar{Q}_c^*$  which implies  $\alpha_1 < Q_{c,1}^* < \alpha_2$ . Hence,  $F(Q_{c,1}^*) \neq 0$  or  $F(Q_{c,1}^*) \neq 1$ .

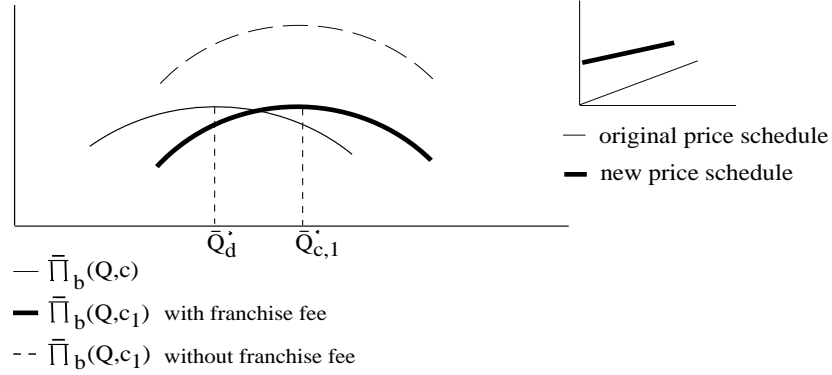
Since  $Q_{c,1}^* \leq \bar{Q}_c^*$ , we have  $F(Q_{c,1}^*) \leq F(\bar{Q}_c^*)$ . Under the new pricing of the vendor, the unit price is  $c_1 = c - (r + b - v) [F(Q_{c,1}^*) - F(\bar{Q}_d^*)]$ . Substituting  $\frac{r+b-c}{r+b-v}$  for the value of  $F(\bar{Q}_d^*)$ , we obtain  $c_1 = r + b - (r + b - v)F(Q_{c,1}^*)$ . Since  $F(Q_{c,1}^*) \leq F(\bar{Q}_c^*)$ , we have  $c_1 > r + b - (r + b - v)F(\bar{Q}_c^*)$ . Recall from Equation (6.4) that  $F(\bar{Q}_c^*) = \frac{r+b-p}{r+b-v}$ . Therefore,  $c_1 > p$  and, hence,  $c_1 > v$  (If  $c < v$  in the Newsboy Problem, the buyer will want to buy an infinite amount). Under the unit price  $c_1$ , it can be shown from Equation (6.2) that  $\bar{Q}_d^*(c_1) = Q_{c,1}^*$ . Therefore, the buyer is motivated to order  $Q_{c,1}^*$  units.

From Proposition 14, we have that  $\bar{\Pi}_b(Q_{c,1}^*, c_1) > \bar{\Pi}_b(\bar{Q}_d^*, c)$ . If the buyer is asked to pay a franchise fee of  $\bar{\Pi}_b(Q_{c,1}^*, c_1) - \bar{\Pi}_b(\bar{Q}_d^*, c)$ , his/her total expected profit is  $\bar{\Pi}_b(Q_{c,1}^*, c_1) - \bar{\Pi}_b(Q_{c,1}^*, c_1) + \bar{\Pi}_b(\bar{Q}_d^*, c) = \bar{\Pi}_b(\bar{Q}_d^*, c)$ . ■

We call the above coordination mechanism the “two-part tariff schedule with fixed cost.” Figure 15 illustrates the effects of the discounted price on the buyer’s expected profits with, and without, the franchise fee. The dashed curve represents  $\bar{\Pi}_b(Q)$  under the discounted price. As seen from the figure, the maximizer of this curve is  $Q_{c,1}^*$ . Therefore, the discount encourages the buyer to order the centralized quantity. However, as formally stated in Proposition 14, the buyer’s expected profit under the discounted price is more than that in the decentralized solution under the original price. Therefore, the profit maximizing vendor, who wants to keep the buyer in a “no worse” situation, charges him/her a fixed payment that results in the dark curve in Figure 15. Note that this kind of a schedule exhibits a decreasing marginal

price.

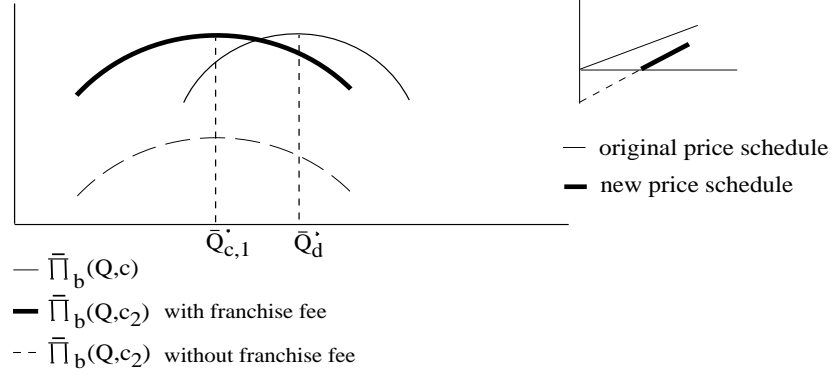
Figure 15 First Coordination Mechanism when  $Q_{c,1}^* > \bar{Q}_d^*$



**PROPOSITION 16** Let  $\Delta_2 = (r + b - v) [F(\bar{Q}_d^*) - F(Q_{c,1}^*)]$  and  $c_2 = c + \Delta_2$ . If  $Q_{c,1}^* < \bar{Q}_d^*$  and  $Q_{c,1}^* > \alpha_1$ , under a unit price increase of  $\Delta_2$  and a franchise fee of  $\bar{\Pi}_b(\bar{Q}_d^*, c) - \bar{\Pi}_b(Q_{c,1}^*, c_2)$  paid by the vendor to the buyer, the buyer stays in a “no worse” situation by ordering  $Q_{c,1}^*$  units.

**Proof:** Noting that  $c_2 = c + (r + b - v) [F(\bar{Q}_d^*) - F(Q_{c,1}^*)]$ , and using Expression (6.2), we obtain  $c_2 = r + b - (r + b - v)F(Q_{c,1}^*)$ . Again using Equation (6.2), we conclude that the buyer can be motivated to order  $Q_{c,1}^*$  units under this new price schedule if the vendor pays the buyer a franchise fee of  $\bar{\Pi}_b(\bar{Q}_d^*, c) - \bar{\Pi}_b(Q_{c,1}^*, c_2)$ . This is simply because, from Proposition 14, we have  $\bar{\Pi}_b(\bar{Q}_d^*, c) > \bar{\Pi}_b(Q_{c,1}^*, c_2)$  so the buyer’s resulting expected total profit amounts to  $\bar{\Pi}_b(\bar{Q}_d^*, c) - \bar{\Pi}_b(Q_{c,1}^*, c_2) + \bar{\Pi}_b(Q_{c,1}^*, c_2) = \bar{\Pi}_b(\bar{Q}_d^*, c)$ . ■

Since the buyer is rewarded for his/her increased expenses, we call the coordination mechanism stated in Proposition 16 and illustrated in Figure 16 the “two-part tariff schedule with fixed reward.” Note that this schedule exhibits an increasing marginal price.

Figure 16 First Coordination Mechanism when  $Q_{c,1}^* < \bar{Q}_d^*$ 

The pricing schedules given in Propositions 15 and 16 coordinate the system in such a way that the buyer orders the centralized order quantity and his/her expected profits are no worse than he/she would otherwise earn. Although this structure of a pricing schedule that coordinates the channel is very simple, the required discount amount may be undesirably large for the vendor. Additionally, if the vendor expects a minimum revenue from the sale of each item, then the maximum discount that can be offered to the buyer is limited. In order to avoid these pitfalls, we next propose a different coordination mechanism in Proposition 17 and 18. In contrast to the previous coordination mechanism, the discount amount needed here to influence the order size of the buyer is smaller.

**PROPOSITION 17** Let  $\Delta_3 = \frac{\bar{\Pi}_b(\bar{Q}_d^*, c) - \bar{\Pi}_b(Q_{c,1}^*, c)}{Q_{c,1}^*}$  and  $c_3 = c - \Delta_3$ . If  $Q_{c,1}^* > \bar{Q}_d^*$ , under a unit discount of  $\Delta_3$  for order sizes greater than or equal to  $Q_{c,1}^*$ ,  $Q_{c,1}^*$  maximizes the buyer's expected profit function. Furthermore,  $\bar{\Pi}_b(Q_{c,1}^*, c_3) = \bar{\Pi}_b(\bar{Q}_d^*, c)$ .

**Proof:** The discounted unit wholesale price is  $c_3 = c - \frac{\bar{\Pi}_b(\bar{Q}_d^*, c) - \bar{\Pi}_b(Q_{c,1}^*, c)}{Q_{c,1}^*}$ . First, we will show that  $\bar{\Pi}_b(Q_{c,1}^*, c_3) = \bar{\Pi}_b(\bar{Q}_d^*, c)$ . Recall that  $\bar{\Pi}_b(Q, c)$  is given by the following

expression.

$$\bar{\Pi}_b(Q, c) = (-c + v)Q - K_b + (r - v)E[X] - (r - v + b) \int_Q^\infty (x - Q)f(x)dx$$

As a consequence, we can write

$$\begin{aligned} \bar{\Pi}_b(Q_{c,1}^*, c_3) &= \left( -c + \frac{\bar{\Pi}_b(\bar{Q}_d^*, c) - \bar{\Pi}_b(Q_{c,1}^*, c)}{Q_{c,1}^*} + v \right) Q_{c,1}^* - K_b + (r - v)E[X] \\ &\quad - (r - v + b) \int_{Q_{c,1}^*}^\infty (x - Q_{c,1}^*)f(x)dx \end{aligned}$$

which in turn leads to

$$\begin{aligned} \bar{\Pi}_b(Q_{c,1}^*, c_3) &= \bar{\Pi}_b(\bar{Q}_d^*, c) - \bar{\Pi}_b(Q_{c,1}^*, c) + (-c + v)Q_{c,1}^* - K_b + (r - v)E[X] \\ &\quad - (r - v + b) \int_{Q_{c,1}^*}^\infty (x - Q_{c,1}^*)f(x)dx \end{aligned}$$

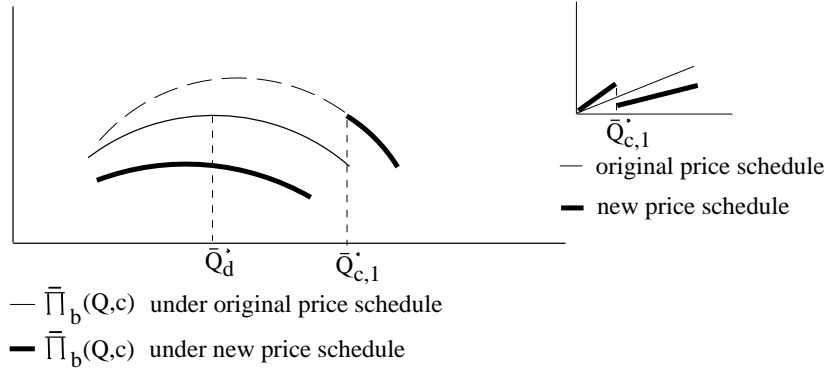
Substituting the expression of  $\bar{\Pi}_b(Q_{c,1}^*, c)$ , the above equation reduces to  $\bar{\Pi}_b(Q_{c,1}^*, c_3) = \bar{\Pi}_b(\bar{Q}_d^*, c)$ .

In order to prove that  $Q_{c,1}^*$  maximizes the buyer's expected profit function, first we will show that  $\bar{Q}_d^* < \bar{Q}_d^*(c_3) < Q_{c,1}^*$ . Since  $c_3 < c$ , it follows from Expression (6.2) that  $\bar{Q}_d^* < \bar{Q}_d^*(c_3)$ . Note that  $\bar{\Pi}_b(\bar{Q}_d^*, c_3) > \bar{\Pi}_b(\bar{Q}_d^*, c)$  because  $c_3 < c$  and  $\bar{\Pi}_b(Q, c)$  is decreasing in  $c$  for fixed values of  $Q$ . Since  $\bar{\Pi}_b(Q_{c,1}^*, c_3) = \bar{\Pi}_b(\bar{Q}_d^*, c)$ , we also have  $\bar{\Pi}_b(\bar{Q}_d^*, c_3) > \bar{\Pi}_b(Q_{c,1}^*, c_3)$ . Recall that for a fixed value of  $c$ ,  $\bar{\Pi}_b(Q, c)$  is a strictly concave function of  $Q$ . Therefore, if  $Q_{c,1}^* > \bar{Q}_d^*$  and  $\bar{Q}_d^* < \bar{Q}_d^*(c_3)$ ,  $\bar{\Pi}_b(\bar{Q}_d^*, c_3) > \bar{\Pi}_b(Q_{c,1}^*, c_3)$  is true only if  $\bar{Q}_d^* < \bar{Q}_d^*(c_3) < Q_{c,1}^*$ . This implies that  $\forall Q > Q_{c,1}^*$  we have  $\bar{\Pi}_b(Q, c_3) < \bar{\Pi}_b(Q_{c,1}^*, c_3)$ . Since  $\bar{Q}_d^*$  maximizes  $\bar{\Pi}_b(Q, c)$ , we have  $\bar{\Pi}_b(Q, c) < \bar{\Pi}_b(\bar{Q}_d^*, c) = \bar{\Pi}_b(Q_{c,1}^*, c_3)$ ,  $\forall Q < Q_{c,1}^*$ , and  $Q \neq \bar{Q}_d^*$ . Therefore,  $Q_{c,1}^*$  maximizes the expected profit function  $\bar{\Pi}_b(Q, c_3)$  under the new pricing schedule. ■

The above coordination mechanism changes the price only after  $Q_{c,1}^*$ . Therefore, the expected profit of the buyer at  $\bar{Q}_d^*$  stays the same. This implies that the buyer is

indifferent to a choice between  $Q_{c,1}^*$  and  $\bar{Q}_d^*$ . However, by slightly increasing the price for order sizes less than  $Q_{c,1}^*$ , the vendor can change the behavior of the buyer so that the buyer orders  $Q_{c,1}^*$  units. The dashed curve in Figure 17 shows how  $\bar{\Pi}_b(Q)$  would appear under the discounted price without any price breaks. However, as seen in Figure 17, in this case, the buyer's expected profits would be maximized at a quantity between  $\bar{Q}_d^*$  and  $Q_{c,1}^*$ . The price breakpoint that the vendor offers, encourages the buyer not to order this quantity. The dark continuous line in Figure 17 shows the buyer's expected profits after a slightly increased unit price before  $Q_{c,1}^*$  and a discount after  $Q_{c,1}^*$ . Since the discount is valid on all items for order sizes greater than or equal to  $Q_{c,1}^*$ , we call this pricing schedule the “all-unit quantity pricing with economies of scale.”

Figure 17 Second Coordination Mechanism when  $Q_{c,1}^* > \bar{Q}_d^*$



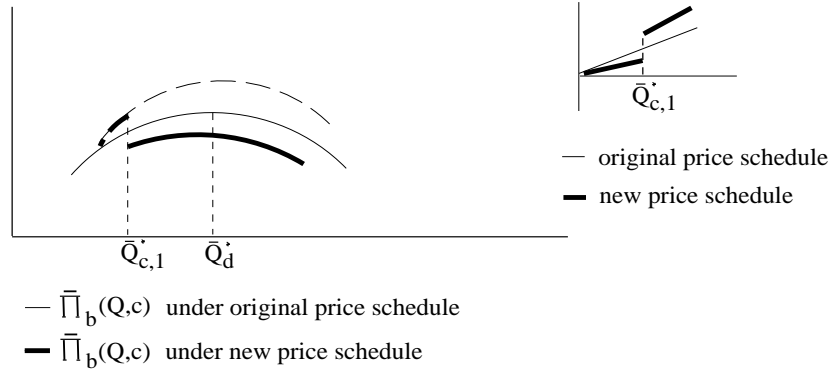
**PROPOSITION 18** Let  $\Delta_4 = \frac{\bar{\Pi}_b(\bar{Q}_d^*, c) - \bar{\Pi}_b(Q_{c,1}^*, c)}{Q_{c,1}^*}$  and  $c_4 = c - \Delta_4$ . If  $Q_{c,1}^* < \bar{Q}_d^*$ , under a unit discount of  $\Delta_4$  for order sizes less than  $Q_{c,1}^*$ ,  $Q_{c,1}^*$  maximizes the buyer's expected profit function. Furthermore,  $\bar{\Pi}_b(Q_{c,1}^*, c_4) = \bar{\Pi}_b(\bar{Q}_d^*, c)$ .

**Proof:** The discounted unit wholesale price is  $c_4 = c - \frac{\bar{\Pi}_b(\bar{Q}_d^*, c) - \bar{\Pi}_b(Q_{c,1}^*, c)}{Q_{c,1}^*}$ . Similarly to the proof of Proposition 17, it can be shown that  $\bar{\Pi}_b(Q_{c,1}^*, c_4) = \bar{\Pi}_b(\bar{Q}_d^*, c)$ . Since

$c_4 < c$ , it follows from Expression (6.2) that  $\bar{Q}_d^* < \bar{Q}_d^*(c_4)$ . Since  $Q_{c,1}^* < \bar{Q}_d^*$ , we have  $\bar{Q}_d^*(c_4) > \bar{Q}_d^* > Q_{c,1}^*$ . Recall that  $\bar{\Pi}_b(Q, c_4)$  is a strictly concave function of  $Q$  and  $\bar{Q}_d^*(c_4)$  is its unique maximizer. It follows that  $\bar{\Pi}_b(Q_{c,1}^*, c_4) > \bar{\Pi}_b(Q, c_4)$ ,  $\forall Q < Q_{c,1}^*$ . For  $Q > Q_{c,1}^*$  and  $Q \neq \bar{Q}_d^*$ , we have  $\bar{\Pi}_b(Q_{c,1}^*, c_4) = \bar{\Pi}_b(\bar{Q}_d^*, c) > \bar{\Pi}_b(Q, c)$ . Therefore,  $Q_{c,1}^*$  maximizes the expected profit function  $\bar{\Pi}_b(Q, c_4)$  under the new pricing schedule. ■

After this coordination mechanism, the buyer is again indifferent to a choice between  $Q_{c,1}^*$  and  $\bar{Q}_d^*$ . However, by slightly increasing the unit price for order sizes greater than  $Q_{c,1}^*$ , the vendor can again influence the behavior of the buyer so that he/she orders  $Q_{c,1}^*$  units (see Figure 18). We call this coordination mechanism the “all-unit quantity pricing with diseconomies of scale.”

Figure 18 Second Coordination Mechanism when  $Q_{c,1}^* < \bar{Q}_d^*$



#### VI.2.4. Decentralized and Centralized Decision Problems for Model II

In the second model, we consider the generalized replenishment cost structure for the buyer as well. The buyer's subproblem is to maximize  $\Pi_b(Q) = \bar{\Pi}_b(Q) - \left\lceil \frac{Q}{P_b} \right\rceil R_b$  over all  $Q > 0$ . As described in Section VI.1,  $\bar{\Pi}_b(Q)$  is a strictly concave function

of  $Q$  with a maximizer at  $\bar{Q}_d^*$ . Note that maximizing  $\Pi_b(Q) = \bar{\Pi}_b(Q) - \left\lceil \frac{Q}{P_b} \right\rceil R_b$  is equivalent to minimizing  $-\Pi_b(Q) = -\bar{\Pi}_b(Q) + \left\lceil \frac{Q}{P_b} \right\rceil R_b$ . Expression (6.8) can again be used to minimize this function by taking  $g(Q) = -\bar{\Pi}_b(Q)$  and  $q = \bar{Q}_d^*$ . Hence,

$$Q_{d,2}^* = \begin{cases} iP_b & \text{if } \mathcal{F} \neq \emptyset, \\ \arg \max \left\{ \Pi_b(\bar{Q}_d^*), \Pi_b \left( \left( \left\lceil \frac{\bar{Q}_d^*}{P_b} \right\rceil - 1 \right) P_b \right) \right\} & \text{if } \mathcal{F} = \emptyset. \end{cases} \quad (6.28)$$

where  $\mathcal{F} = \{k \in Z^+ : -\bar{\Pi}_b(kP_b) + \bar{\Pi}_b((k+1)P_b) < R_b, (k+1)P_b \leq \bar{Q}_d^*\}$  and  $i = \min\{k \text{ s.t. } k \in \mathcal{F}\}$  when  $\mathcal{F} \neq \emptyset$ .

In the centralized solution, we maximize  $\Pi_c^H(Q) = \Pi_v(Q) + \Pi_b(Q)$ . Note that  $\Pi_v(Q) = \bar{\Pi}_v(Q) - \left\lceil \frac{Q}{P_v} \right\rceil R_v$  and  $\Pi_b(Q) = \bar{\Pi}_b(Q) - \left\lceil \frac{Q}{P_b} \right\rceil R_b$ . Therefore,  $\Pi_c^H(Q)$  can be rewritten as

$$\Pi_c^H(Q) = \bar{\Pi}_v(Q) + \bar{\Pi}_b(Q) - \left\lceil \frac{Q}{P_v} \right\rceil R_v - \left\lceil \frac{Q}{P_b} \right\rceil R_b$$

Note also that  $\bar{\Pi}_v(Q) + \bar{\Pi}_b(Q) = \bar{\Pi}_c(Q)$ . This leads to

$$\Pi_c^H(Q) = \bar{\Pi}_c(Q) - \left\lceil \frac{Q}{P_v} \right\rceil R_v - \left\lceil \frac{Q}{P_b} \right\rceil R_b \quad (6.29)$$

Recall that  $\bar{\Pi}_c(Q)$  is the expected system profits of the centralized solution when no truck costs or capacity are included. It is a concave function of  $Q$  with a maximizer at  $\bar{Q}_c^*$ .

Based on the following properties of  $\Pi_c^H(Q)$ , we provide a finite time exact solution procedure for its maximization.

**PROPERTY 19** Let  $Q_2 > Q_1 > \bar{Q}_c^*$ . Then  $\Pi_c^H(Q_2) < \Pi_c^H(Q_1)$ . That is,  $\Pi_c^H(Q)$  is decreasing after  $\bar{Q}_c^*$ .

**Proof:** Since  $\bar{\Pi}_c(Q)$  is a strictly concave function of  $Q$  and  $\bar{Q}_c^*$  is its maximizer,  $\bar{\Pi}_c(Q)$  is decreasing  $\forall Q > \bar{Q}_c^*$ . Observe that  $(-\lceil Q/P_v \rceil R_v)$  and  $(-\lceil Q/P_b \rceil R_b)$  are

nonincreasing functions. The sum of a decreasing and a nonincreasing function is decreasing. Therefore,  $\Pi_c^{II}(Q)$  is decreasing  $\forall Q > \bar{Q}_c^*$ . ■

**PROPERTY 20** Let  $Q_1$  and  $Q_2$  be such that  $(k_1 - 1)P_b < Q_1 < Q_2 \leq k_1 P_b \leq \bar{Q}_c^*$  and  $(k_2 - 1)P_v < Q_1 < Q_2 \leq k_2 P_v \leq \bar{Q}_c^*$  where  $k_1 \in Z^+$  and  $k_2 \in Z^+$ . Then  $\Pi_c^{II}(Q_1) < \Pi_c^{II}(Q_2)$ . In other words for  $Q \leq \bar{Q}_c^*$ ,  $\Pi_c^{II}(Q)$  is piece-wise increasing.

**Proof:** Since  $\bar{\Pi}_c(Q)$  is concave with a maximizer at  $\bar{Q}_c^*$ ,  $\bar{\Pi}_c(Q_1) < \bar{\Pi}_c(Q_2) \forall Q_1, Q_2$  such that  $Q_1 < Q_2 \leq \bar{Q}_c^*$ . When  $(k_1 - 1)P_b < Q_1 < Q_2 \leq k_1 P_b$ , we have  $\lceil Q_1/P_b \rceil = \lceil Q_2/P_b \rceil = k_1$ . Similarly, when  $(k_2 - 1)P_v < Q_1 < Q_2 \leq k_2 P_v$ , we have  $\lceil Q_1/P_v \rceil = \lceil Q_2/P_v \rceil = k_2$ . Therefore,  $\bar{\Pi}_c(Q_1) - \lceil Q_1/P_v \rceil R_v - \lceil Q_1/P_b \rceil R_b < \bar{\Pi}_c(Q_2) - \lceil Q_2/P_v \rceil R_v - \lceil Q_2/P_b \rceil R_b$  so that  $\Pi_c^{II}(Q_1) < \Pi_c^{II}(Q_2)$ . ■

Therefore, in computing  $\Pi_c^{II}(Q_2)$ , we need to consider  $\bar{Q}_c^*$  and the integer multiples of  $P_b$  and  $P_v$  that are less than or equal to  $\bar{Q}_c^*$ .

### VI.2.5. Coordinated Solution for Model II

**PROPOSITION 19** If  $Q_{c,2}^* > Q_{d,2}^*$ , the following coordination mechanism maximizes the buyer's expected profit function with a maximum function value of  $\Pi_b(Q_{d,2}^*)$  at  $Q_{c,2}^*$ .

- If  $Q_{c,2}^* > \bar{Q}_d^*$ , the vendor pays the buyer a fixed franchise fee of  $\Pi_b(Q_{d,2}^*) - \Pi_b(Q_{c,2}^*)$  for orders larger than or equal to  $Q_{c,2}^*$ .
- If  $Q_{c,2}^* < \bar{Q}_d^*$ , the vendor pays the buyer a fixed franchise fee of  $\Pi_b(Q_{d,2}^*) - \Pi_b(Q_{c,2}^*)$  for order sizes in the range  $((\lceil Q_{c,2}^*/P_b \rceil - 1)P_b, Q_{c,2}^*]$

**Proof:** Note that in both cases the buyer is compensated with a fixed payment of  $\Pi_b(Q_{d,2}^*) - \Pi_b(Q_{c,2}^*)$ . Therefore, if he/she orders  $Q_{c,2}^*$ , then he/she stays at a “no worse” expected profit value which is  $\Pi_b(Q_{d,2}^*) - \Pi_b(Q_{c,2}^*) + \Pi_b(Q_{c,2}^*) = \Pi_b(Q_{d,2}^*)$ . We



next show that in each case the maximum attainable expected profit value for the buyer is not greater than  $\Pi_b(Q_{d,2}^*)$ .

For the first part (i.e.,  $Q_{c,2}^* > \bar{Q}_d^*$ ), we use Proposition 10 which implies that  $\Pi_b(Q)$  is decreasing after  $\bar{Q}_d^*$ . That is,  $\Pi_b(Q_2) < \Pi_b(Q_1)$  for all  $Q_1$  and  $Q_2$  such that  $\bar{Q}_d^* \leq Q_1 < Q_2$ . Adding a fixed value of  $\Pi_b(Q_{d,2}^*) - \Pi_b(Q_{c,2}^*)$  to both sides of this inequality results in  $\Pi_b(Q_2) + \Pi_b(Q_{d,2}^*) - \Pi_b(Q_{c,2}^*) < \Pi_b(Q_1) + \Pi_b(Q_{d,2}^*) - \Pi_b(Q_{c,2}^*)$  which implies that the buyer's expected profit function is still decreasing after  $\bar{Q}_d^*$  under the new pricing strategy. Therefore, the maximum value of the buyer's expected profit in this region is realized at the smallest value of  $Q$  which is  $Q_{c,2}^*$ .

For the second part (i.e.,  $Q_{c,2}^* < \bar{Q}_d^*$ ), we use Proposition 11 which implies that to the left of  $\bar{Q}_d^*$ ,  $\Pi_b(Q)$  is piecewise increasing. Adding a fixed value to each piece does not change the fact that  $\Pi_b(Q)$  is increasing in  $(\lceil Q_{c,2}^*/P_b \rceil - 1)P_b, Q_{c,2}^*]$ . Therefore, under the new pricing strategy,  $Q_{c,2}^*$  maximizes the buyer's expected profit. ■

**PROPOSITION 20** If  $Q_{c,2}^* < Q_{d,2}^*$ , the following coordination mechanism maximizes the buyer's expected profit function with a maximum function value of  $\Pi_b(Q_{d,2}^*)$  at  $Q_{c,2}^*$ .

- If  $Q_{c,2}^* = kP_b$  for some positive integer  $k$ , then the vendor pays the buyer a franchise fee of  $\Pi_b(Q_{d,2}^*) - \Pi_b(Q_{c,2}^*)$  for order sizes less than or equal to  $Q_{c,2}^*$ .
- If  $Q_{c,2}^* = kP_v$ , the vendor pays the buyer a fixed franchise fee of  $\Pi_b(Q_{d,2}^*) - \Pi_b(Q_{c,2}^*)$  for order sizes in the range  $(\lceil Q_{c,2}^*/P_b \rceil - 1)P_b, Q_{c,2}^*]$ .

**Proof:** From Expression (6.28), we have  $Q_{d,2}^* \leq \bar{Q}_d^*$ . We also know from Equations (6.2) and (6.4) that  $\bar{Q}_d^* \leq \bar{Q}_c^*$ . Therefore,  $Q_{d,2}^* \leq \bar{Q}_d^* \leq \bar{Q}_c^*$  which implies that  $Q_{c,2}^* < \bar{Q}_d^*$  and  $Q_{c,2}^* < \bar{Q}_c^*$ . As a result,  $Q_{c,2}^*$  can either be an integer multiple of  $P_b$  or  $P_v$ . In both cases, if the buyer is compensated with a fixed payment of  $\Pi_b(Q_{d,2}^*) - \Pi_b(Q_{c,2}^*)$

and orders  $Q_{c,2}^*$  units, then he/she again stays at a “no worse” than expected profit level which is  $\Pi_b(Q_{d,2}^*) - \Pi_b(Q_{c,2}^*) + \Pi_b(Q_{c,2}^*) = \Pi_b(Q_{d,2}^*)$ . However, in order to complete the proof, we need to show that the maximum attainable expected profit value for the buyer is not greater than  $\Pi_b(Q_{d,2}^*)$ .

From Property 11, we know that for  $Q < \bar{Q}_d^*$ ,  $\Pi_b(Q)$  is piecewise increasing. Since  $Q_{c,2}^* < Q_{d,2}^*$ , we can also conclude that before  $Q_{c,2}^*$ ,  $\Pi_b(Q)$  is increasing at integer multiples of  $P_b$ . Otherwise, from Property 12,  $Q_{d,2}^*$  would take a value that is less than  $Q_{c,2}^*$ . Therefore, for the first part of the proposition, we show that the buyer’s expected profit function in the given region (i.e.  $(-\infty, Q_{c,2}^*]$ ) takes its maximum at  $Q_{c,2}^*$ . The second part of the proof follows from the fact that  $\Pi_b(Q)$  is piecewise increasing. ■

### VI.3. Summary

In this chapter, we considered the channel coordination problem in the Newsboy setting. As discussed in Chapter II, no existing study in the literature appears to investigate the channel coordination problem with transportation costs and capacity incorporated. Although there are a few studies that consider stochasticity of demand in channel coordination problems, this again is an issue that needs further attention. We believe that one of the contributions of this chapter is the consideration of these two issues.

Similar to the results in Chapter IV, we again showed that when the vendor has a generalized replenishment cost structure, there may be cases where it is better for him/her to arrange for smaller order quantities from the buyer. We characterized the conditions under which this situation occurs. We also observed that for single period problems, this case is more common than for infinite horizon problems, specifically

the ones we considered in Chapters III and IV. We believe the reason for this is that, due to the recurrent nature of infinite horizon models, there is an opportunity to reduce the truck costs by increasing the order sizes and hence take advantage of full truck loads. In this way, additional savings from less than full truck loads in future periods can be achieved. On the other hand, in single period models, this opportunity is limited, and there is no opportunity to save from future period costs by ordering more. Therefore, we believe that transportation costs and capacities have more effect on channel coordination issues in single period problems. In this chapter, we also introduced four efficient new mechanisms for channel coordination into the literature.

## CHAPTER VII

### BUYER-VENDOR PROBLEM UNDER DEPRECIATING ECONOMIC VALUE OF ITEMS

In Chapters III, IV, V and VI of the dissertation, we focused mainly on coordination issues with transportation capacity and cost considerations. We showed various interesting results pertaining to theory and practice for the buyer-vendor coordination problem. Our models in Chapters III, IV and V assumed deterministic demand, whereas the ones in Chapter VI assumed random demand. As we pointed out earlier in Chapter II, stochastic channel coordination studies are very limited in the literature. Also, stochastic models are better representations of real life. In this chapter, we investigate the channel coordination problem in another setting that takes into account the stochastic nature of demand. In fact, we not only consider this practical situation, but unlike in the other chapters, we take retail price as a decreasing function of time. This is a common situation in the retail industry, especially for items with short product life cycles such as consumer electronics or fashion items. Therefore, as in Chapter VI, we investigate the problem in the Newsboy setting, which considers the replenishment decisions in a single period representing the life-cycle of the product.

The importance of this kind of a retail price structure for channel coordination is that, since the buyer's marginal profit from a unit item decreases over time, fixed discounts proposed by the vendor may not be an efficient coordination mechanism. In fact, we show here that the discount value for coordinating the channel depends on the life-cycle length of the product. We propose effective coordination mechanisms which consider life-cycle length of the product as part of the negotiation mechanism or as a decision variable.

In Section VII.1, we consider the simpler case where demand is not affected by the retail price. That is, customers come to the store according to a fixed rate at all times. More specifically, we model the demand arrival process as a pure Poisson Process. In Section VII.2, we consider the case where more customers are willing to buy the item when it is cheaper; that is, the demand depends on the retail price. In order to model this situation, we take the demand arrival process as a Non-homogenous Poisson Process where the arrival rate is a function of retail price, which itself depends on time.

The notation we use in this chapter is essentially the same as in Chapter VI. However, here we also have a per unit, per unit time inventory holding cost ( $h_b$ ) at the buyer. It is important to note that most single period stochastic demand problems do not model inventory holding costs. This is because, either they assume that the period is so short that these costs can be ignored, or they charge the inventory holding costs to the end-of-period items (i.e. unsold items) by modifying the salvage value. In this sense, our analysis is more exact, because we charge an inventory holding cost for each time unit that an item stays in the inventory.

We introduce the following additional notation.

$T$ : Period length.

$\alpha$ : Selling price of the item at time 0.

$\beta$ : Rate of depreciation in the economical value of an item ( $\alpha - \beta T > 0$ ).

$S_i$ : Arrival time of the  $i^{th}$  demand.

$N(T)$ : Number of demand arrivals during  $[0, T]$ .

We believe that the models we present in this chapter can be extended to consider the length of the life-cycle, i.e., planning horizon, as a decision variable. Hence, we explicitly use  $T$  to represent it.

## VII.1. Single Period, Single Replenishment Model with Time Dependent Retail Price

In addition to the classical assumptions of the Newsboy Model at the buyer, we assume that the demand arrival process is a pure Poisson Process with rate  $\lambda$ . We consider the case where the retail price of an item depends on the time it is sold. That is,  $r(t) = \alpha - \beta t$ . We first model and solve the problem using the decentralized approach.

### VII.1.1. Decentralized Model

#### Buyer's Subproblem

The buyer's income consists of his/her revenue from regular sales and the salvage value of any remaining items at the end of period  $T$ . The expenses that the buyer incurs are inventory holding cost, lost sale cost, purchase cost and replenishment cost. Therefore, the expected value of the buyer's profits as a function of his/her order quantity is given by:

$$\begin{aligned}\Pi_b(Q) = & E[\text{Revenue}] + E[\text{Salvage Value}] \\ & - E[\text{Holding cost}] - E[\text{Lost sale cost}] \\ & - E[\text{Purchase cost}] - E[\text{Replenishment cost}]\end{aligned}\tag{7.1}$$

In calculating the terms of the above expression, some properties of order statistics will be used. These properties will be presented below in Theorems 6, 7 and Proposition 21. But, we first provide the following formal definition of order statistics.

**DEFINITION 1** Let  $\{X_1, X_2, \dots, X_n\}$  be an independent set of identically dis-

tributed continuous random variables with common density and distribution functions  $f(t)$  and  $F(t)$ , respectively. Let  $Y_1 = \min\{X_1, X_2, \dots, X_n\}$ ,  $Y_n = \max\{X_1, X_2, \dots, X_n\}$ , and in general,  $Y_k$  ( $1 \leq k \leq n$ ) be the  $k^{th}$  smallest value in  $\{X_1, X_2, \dots, X_n\}$ . Then,  $Y_k$  is called the  $k^{th}$  order statistic, and the set  $\{Y_1, Y_2, \dots, Y_n\}$  is said to consist of the order statistics of  $\{X_1, X_2, \dots, X_n\}$  (see Ghahramani (2000), page 345).

**THEOREM 6** Let  $\{Y_1, Y_2, \dots, Y_n\}$  be the order statistics of the independent identically distributed continuous random variables  $X_1, X_2, \dots, X_n$  with the common probability distribution and probability density functions  $F(t)$  and  $f(t)$ , respectively. Then  $F_{Y_k}(t)$  and  $f_{Y_k}(t)$ , the probability distribution and probability density functions of  $Y_k$ , respectively, are given by

$$F_{Y_k}(t) = \sum_{i=k}^n \binom{n}{i} [F(t)]^i [1 - F(t)]^{n-i}, \quad -\infty < t < \infty,$$

and

$$f_{Y_k}(t) = \frac{n!}{(k-1)!(n-k)!} F(t)^{k-1} f(t) (1 - F(t))^{(n-k)}, \quad -\infty < t < \infty. \quad (7.2)$$

**Proof:** See Ghahramani (2000), page 346.

**THEOREM 7** Let  $\{N(t), t \geq 0\}$  be a Poisson Process with rate parameter  $\lambda$ , and let  $S_n$  be the  $n^{th}$  event time. Given  $N(t) = n$

$$(S_1, S_2, \dots, S_n) \stackrel{d}{=} (\bar{U}_1, \bar{U}_2, \dots, \bar{U}_n)$$

where  $\bar{U}_1, \bar{U}_2, \dots, \bar{U}_n$  are the order statistics of  $n$  i.i.d. random variables  $U_1, U_2, \dots, U_n$  distributed uniformly over  $[0, t]$  and  $\stackrel{d}{=}$  denotes in distribution.

**Proof:** See Kulkarni (1995), page 209.

**PROPOSITION 21** Let  $U_1, U_2, \dots, U_n$  be i.i.d. random variables uniformly distributed over  $[0, t]$  and let  $\bar{U}_1, \bar{U}_2, \dots, \bar{U}_n$  be the order statistics of  $U_1, U_2, \dots, U_n$ . Then,

$$E(\bar{U}_k) = \frac{kt}{n+1}, \quad 1 \leq k \leq n.$$

**Proof:**

Utilizing Expression (7.2), we have the density function for the  $k^{th}$  order statistics of  $n$  uniformly distributed random variables over  $[0, t]$  as follows:

$$f_{\bar{U}_k}(u) = \frac{n!}{(k-1)!(n-k)!} \left(\frac{u}{t}\right)^{k-1} \frac{1}{t} \left(1 - \frac{u}{t}\right)^{n-k}.$$

Since  $U_1, U_2, \dots, U_n$  are uniformly distributed over  $[0, t]$ ;  $\bar{U}_1, \bar{U}_2, \dots, \bar{U}_n$  also take values within  $[0, t]$ . This leads to

$$E[\bar{U}_k] = \int_0^t u \frac{n!}{(k-1)!(n-k)!} \left(\frac{u}{t}\right)^{k-1} \frac{1}{t} \left(1 - \frac{u}{t}\right)^{n-k} du.$$

Taking the constant terms out of the integral, the above expression can be written as

$$E[\bar{U}_k] = \frac{n!}{(k-1)!(n-k)!} \int_0^t \left(\frac{u}{t}\right)^k \left(1 - \frac{u}{t}\right)^{n-k} du.$$

Letting  $x = \frac{u}{t}$ , we have  $du = tdx$ . By making a change in variables, the above expression can be rewritten as

$$E[\bar{U}_k] = \frac{n!}{(k-1)!(n-k)!} \int_0^1 x^k (1-x)^{n-k} t dx.$$

From standard probability laws, we know that

$$\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

where  $\Gamma(\alpha)$  is the so called Gamma function (see Larson (1982), page 206). Note that  $\Gamma(r) = (r-1)!$  where  $r$  is a positive integer (see Larson (1982), page 199). Expressing



$E[\bar{U}_k]$  in terms of the Gamma function leads to

$$E[\bar{U}_k] = \frac{n!t}{(k-1)!(n-k)!} \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)}.$$

Since  $n$  and  $k$  are positive integers such that  $1 \leq k \leq n$ , we have

$$E[\bar{U}_k] = \frac{n!t}{(k-1)!(n-k)!} \frac{k!(n-k)!}{(n+1)!} = \frac{kt}{n+1}.$$

■

Returning to Expression (7.1), we next evaluate each component of this expression. The buyer's revenue from the sale of regular items, the earnings from the salvage value of unsold items, the inventory holding cost and the lost sale cost are all functions of the number of items demanded during a horizon of  $T$  (i.e.  $N(T)$ ), which is a random variable. Therefore, all of these terms are also random variables whose expectations can be computed using conditional expectation.

#### **Expected Revenue of the Buyer:**

$$\begin{aligned} E[\text{Revenue}] &= E[E[\text{Revenue}|N(T)]] \\ &= \sum_{n=0}^{\infty} E[\text{Revenue}|N(T) = n]P\{N(T) = n\} \\ &= E[\text{Revenue}|N(T) = 0]P\{N(T) = 0\} + \\ &\quad \sum_{n=1}^Q E[\text{Revenue}|N(t) = n]P\{N(T) = n\} + \\ &\quad \sum_{n=Q+1}^{\infty} E[\text{Revenue}|N(t) = n]P\{N(T) = n\}. \end{aligned} \tag{7.3}$$

Note that  $E[\text{Revenue}|N(T) = 0] = 0$ .

For  $1 \leq n \leq Q$ ,

$$E[\text{Revenue}|N(T) = n] = E \left[ \sum_{i=1}^{N(T)} (\alpha - \beta S_i) | N(T) = n \right].$$

By using Theorem (7) and the independence of  $N(T)$  and  $S_i$ , we can further write

$$E \left[ \sum_{i=1}^{N(T)} (\alpha - \beta S_i) | N(T) = n \right] = E \left[ \sum_{i=1}^n (\alpha - \beta \bar{U}_i) \right].$$

Observe that  $E [\sum_1^n \bar{U}_i] = E [\sum_1^n U_i]$  because the sum of  $n$  random variables, whether they are ordered or unordered, is the same. This implies that for  $1 \leq n \leq Q$

$$\begin{aligned} E \left[ \sum_{i=1}^n (\alpha - \beta \bar{U}_i) \right] &= \alpha n - \beta E \left[ \sum_{i=1}^n U_i \right] \\ &= \alpha n - \beta n T / 2. \end{aligned}$$

Therefore, for  $1 \leq n \leq Q$

$$E [\text{Revenue} | N(T) = n] = \alpha n - \beta n T / 2. \quad (7.4)$$

For  $Q < n < \infty$ ,

$$E [\text{Revenue} | N(T) = n] = E \left[ \sum_{i=1}^Q (\alpha - \beta S_i) | N(T) = n \right]$$

Again, using a similar argument, we can write

$$E \left[ \sum_{i=1}^Q (\alpha - \beta S_i) | N(T) = n \right] = \alpha Q - \beta E \left[ \sum_{i=1}^Q \bar{U}_i \right].$$

Using Proposition 21 we have

$$E \left[ \sum_{i=1}^Q \bar{U}_i \right] = \sum_{i=1}^Q E [\bar{U}_i] = \sum_{i=1}^Q \frac{iT}{n+1} = \frac{Q(Q+1)T}{2(n+1)}.$$

Therefore, for  $Q < n < \infty$

$$E [\text{Revenue} | N(T) = n] = \alpha Q - \beta E \left[ \sum_{i=1}^Q \bar{U}_i \right] = \alpha Q - \beta \frac{Q(Q+1)T}{2(n+1)}. \quad (7.5)$$

By utilizing Expressions (7.4) and (7.5) and the fact that

$$E[\text{Revenue}|N(T) = 0] = 0,$$

the expected revenue of the buyer as expressed in (7.3) can be written as

$$\begin{aligned} E[\text{Revenue}] &= \sum_{n=1}^Q \left\{ \alpha n - \frac{\beta n T}{2} \right\} P\{N(T) = n\} + \\ &\quad \sum_{n=Q+1}^{\infty} \left\{ \alpha Q - \frac{\beta Q(Q+1)T}{2(n+1)} \right\} P\{N(T) = n\}. \end{aligned} \quad (7.6)$$

### Expected Inventory Holding Cost of the Buyer:

The buyer's expected inventory holding cost during the period  $[0, T]$  is again calculated by conditioning on  $N(T)$ .

$$\begin{aligned} E[\text{Holding cost}] &= E[E[\text{Holding cost}|N(T)]] \\ &= \sum_{n=0}^{\infty} E[\text{Holding cost}|N(T) = n] P\{N(T) = n\} \\ &= E[\text{Holding cost}|N(T) = 0] P\{N(T) = 0\} + \\ &\quad \sum_{n=1}^Q E[\text{Holding cost}|N(T) = n] P\{N(T) = n\} + \\ &\quad \sum_{n=Q+1}^{\infty} E[\text{Holding cost}|N(T) = n] P\{N(T) = n\}. \end{aligned} \quad (7.7)$$

When the total demand during  $[0, T]$  is 0, all of the  $Q$  items purchased at the beginning of the period incur a per unit inventory holding cost of  $\$h_b$  for  $T$  units of time. Therefore,

$$E[\text{Holding cost}|N(T) = 0] = Qh_bT.$$

For  $1 \leq n \leq Q$ , an item sold at time  $S_i$  ( $S_i \leq T$ ) incurs a total of  $\$S_i h_b$  holding cost, and each of the end-of-period items incurs a total of  $\$h_b T$  holding cost.

Accordingly,

$$\begin{aligned}
E[\text{Holding cost}|N(T) = n] &= E \left[ \left\{ \sum_{i=1}^{N(T)} S_i h_b + (Q - N(T)) h_b T \right\} | N(T) = n \right] \\
&= E \left[ \sum_{i=1}^n \bar{U}_i h_b + (Q - n) h_b T \right] \\
&= \frac{h_b n T}{2} + (Q - n) h_b T = Q h_b T - \frac{h_b n T}{2}. \tag{7.8}
\end{aligned}$$

For  $Q < n < \infty$ , since all items are demanded during  $[0, T]$ , each of them incurs a total of  $\$S_i h_b$  inventory holding cost. Therefore,

$$\begin{aligned}
E[\text{Holding cost}|N(T) = n] &= E \left[ \sum_{i=1}^Q S_i h_b | N(T) = n \right] \\
&= h_b E \left[ \sum_{i=1}^Q \bar{U}_i \right] \\
&= h_b \frac{Q(Q+1)T}{2(n+1)}. \tag{7.9}
\end{aligned}$$

Using Expressions (7.8) and (7.9) and the fact that  $E[\text{Holding cost}|N(T) = 0] = Q h_b T$ , Expression (7.7) can be rewritten as

$$\begin{aligned}
E[\text{Holding cost}] &= Q h_b T P\{N(T) = 0\} + \\
&\quad \sum_{n=1}^Q \left\{ Q h_b T - \frac{h_b n T}{2} \right\} P\{N(T) = n\} + \\
&\quad \sum_{n=Q+1}^{\infty} \frac{h_b Q(Q+1)T}{2(n+1)} P\{N(T) = n\}. \tag{7.10}
\end{aligned}$$

### Expected Salvage Value of Unsold Items at the Buyer:

Items unsold at the end of the period are salvaged with a per unit earning of  $\$v$ .

Thus,

$$\begin{aligned}
 E[\text{Salvage Value}] &= E[E[\text{Salvage Value}|N(T)]] \\
 &= \sum_{n=0}^Q v(Q-n)P\{N(T)=n\}.
 \end{aligned} \tag{7.11}$$

### Expected Lost Sale Cost of the Buyer:

Each demand that arrives after the first  $Q$  units is lost. The buyer incurs a penalty of  $\$b/\text{unit}$  for these items. Therefore,

$$\begin{aligned}
 E[\text{Lost sale cost}] &= E[E[\text{Lost sale cost}|N(T)]] \\
 &= \sum_{n=Q+1}^{\infty} b(n-Q)P\{N(T)=n\}.
 \end{aligned} \tag{7.12}$$

### Expected Sum of Purchase and Replenishment Costs of the Buyer:

At the beginning of the period, the buyer orders  $Q$  units and pays a total of  $\$cQ$  for purchase costs. Additionally, there is a fixed replenishment cost of  $\$K_b$ . Therefore,

$$E[\text{Purchase cost} + \text{Replenishment cost}] = cQ + K_b. \tag{7.13}$$

We have now calculated all of the terms of Expression (7.1). Using Expressions (7.6), (7.10), (7.11), (7.12) and (7.13),  $\Pi_b(Q)$  can explicitly be written as

$$\begin{aligned}
 \Pi_b(Q) &= \sum_{n=1}^Q \left\{ \alpha n - \frac{\beta n T}{2} \right\} P\{N(T)=n\} \\
 &\quad + \sum_{n=Q+1}^{\infty} \left\{ \alpha Q - \frac{\beta Q(Q+1)T}{2(n+1)} \right\} P\{N(T)=n\} \\
 &\quad + \sum_{n=0}^Q v(Q-n)P\{N(T)=n\} - Qh_bTP\{N(T)=0\} \\
 &\quad - \sum_{n=1}^Q \left\{ Qh_bT - \frac{h_b n T}{2} \right\} P\{N(T)=n\}
 \end{aligned}$$

$$\begin{aligned}
& - \sum_{n=Q+1}^{\infty} \frac{h_b Q(Q+1)T}{2(n+1)} P\{N(T) = n\} \\
& - \sum_{n=Q+1}^{\infty} (n-Q)bP\{N(T) = n\} - cQ - K_b.
\end{aligned}$$

Rearranging the terms of the above expression, we obtain

$$\begin{aligned}
\Pi_b(Q) &= -cQ - K_b + Q(v - h_b T)P\{N(T) = 0\} + \\
& \sum_{n=1}^Q \left\{ \alpha n - \frac{\beta n T}{2} + \frac{h_b n T}{2} - Q h_b T + (Q - n)v \right\} P\{N(T) = n\} + \\
& \sum_{n=Q+1}^{\infty} \left\{ \alpha Q - \frac{\beta Q(Q+1)T}{2(n+1)} - \frac{h_b Q(Q+1)T}{2(n+1)} - (n-Q)b \right\} P\{N(T) = n\}.
\end{aligned} \tag{7.14}$$

Next, we analyze the properties of the  $\Pi_b(Q)$  function.

**PROPOSITION 22**  $\Pi_b(Q)$  in Expression (7.14) is a concave function of  $Q$ .

**Proof:**

Let  $\Delta \Pi_b(Q) = \Pi_b(Q+1) - \Pi_b(Q)$ . Using (7.14), we have

$$\begin{aligned}
\Delta \Pi_b(Q) &= \\
& -c(Q+1) + (Q+1)(v - h_b T)P\{N(T) = 0\} \\
& + \sum_{n=1}^{Q+1} \left\{ \alpha n - \frac{\beta n T}{2} + \frac{h_b n T}{2} - (Q+1)h_b T + (Q+1-n)v \right\} P\{N(T) = n\} \\
& + \sum_{n=Q+2}^{\infty} \left\{ \alpha(Q+1) - \frac{\beta(Q+1)(Q+2)T}{2(n+1)} - \frac{h_b(Q+1)(Q+2)T}{2(n+1)} \right. \\
& \quad \left. - (n-Q-1)b \right\} P\{N(T) = n\} + cQ - Q(v - h_b T)P\{N(T) = 0\} \\
& - \sum_{n=1}^Q \left\{ \alpha n - \frac{\beta n T}{2} + \frac{h_b n T}{2} - Q h_b T + (Q-n)v \right\} P\{N(T) = n\} \\
& - \sum_{n=Q+1}^{\infty} \left\{ \alpha Q - \frac{\beta Q(Q+1)T}{2(n+1)} - \frac{h_b Q(Q+1)T}{2(n+1)} - (n-Q)b \right\} P\{N(T) = n\}.
\end{aligned}$$

After some cancellation and rearrangement of terms, the above expression can be written as

$$\begin{aligned}
\Delta\Pi_b(Q) = & -c + (v - h_b T)P\{N(T) = 0\} + \sum_{n=1}^Q \{-h_b T + v\}P\{N(T) = n\} \\
& + \left\{ \alpha(Q+1) - \frac{\beta(Q+1)T}{2} + \frac{h_b(Q+1)T}{2} - (Q+1)h_b T \right\} P\{N(T) = Q+1\} \\
& + \sum_{n=Q+2}^{\infty} \left\{ \alpha - \frac{\beta(Q+1)T}{(n+1)} - \frac{h_b(Q+1)T}{(n+1)} + b \right\} P\{N(T) = n\} \\
& - \left\{ \alpha Q - \frac{\beta Q(Q+1)T}{2(Q+2)} - \frac{h_b Q(Q+1)}{2(Q+2)} - b \right\} P\{N(T) = Q+1\}.
\end{aligned}$$

Further rearrangement of the terms leads to

$$\begin{aligned}
\Delta\Pi_b(Q) = & -c + \sum_{n=0}^Q \{v - h_b T\}P\{N(T) = n\} \\
& + \left\{ \alpha - \frac{\beta(Q+1)}{Q+2} - \frac{h_b(Q+1)}{Q+2} + b \right\} P\{N(T) = Q+1\} \\
& + \sum_{n=Q+2}^{\infty} \left\{ \alpha - \frac{\beta(Q+1)T}{n+1} - \frac{h_b(Q+1)T}{n+1} + b \right\} P\{N(T) = n\},
\end{aligned}$$

and hence

$$\begin{aligned}
\Delta\Pi_b(Q) = & -c + \sum_{n=0}^Q \{v - h_b T\}P\{N(T) = n\} \\
& + \sum_{Q+1}^{\infty} \left\{ \alpha - \frac{\beta(Q+1)T}{n+1} - \frac{h_b(Q+1)T}{n+1} + b \right\} P\{N(T) = n\}. \quad (7.15)
\end{aligned}$$

Let  $\Delta^2\Pi_b(Q) = \Delta\Pi_b(Q+1) - \Delta\Pi_b(Q)$ . Thus,

$$\begin{aligned}\Delta^2\Pi_b(Q) &= \sum_{n=0}^{Q+1} \{v - h_b T\} P\{N(T) = n\} \\ &\quad + \sum_{n=Q+2}^{\infty} \left\{ \alpha - \frac{\beta(Q+2)T}{n+1} - \frac{h_b(Q+2)T}{n+1} + b \right\} P\{N(T) = n\} \\ &\quad - \sum_{n=0}^Q \{v - h_b T\} P\{N(T) = n\} \\ &\quad - \sum_{n=Q+1}^{\infty} \left\{ \alpha - \frac{\beta(Q+1)T}{n+1} - \frac{h_b(Q+1)T}{n+1} + b \right\} P\{N(T) = n\},\end{aligned}$$

which can further be reduced to

$$\begin{aligned}\Delta^2\Pi_b(Q) &= \{v - h_b T\} P\{N(T) = Q+1\} \\ &\quad + \sum_{n=Q+2}^{\infty} \left\{ -\frac{\beta T}{n+1} - \frac{h_b T}{n+1} \right\} P\{N(T) = n\} \\ &\quad - \left\{ \alpha - \frac{\beta(Q+1)T}{Q+2} - \frac{h_b(Q+1)T}{Q+2} + b \right\} P\{N(T) = Q+1\}.\end{aligned}$$

It follows that

$$\begin{aligned}\Delta^2\Pi_b(Q) &= \left\{ v - b - \alpha + \frac{\beta(Q+1)T}{Q+2} - \frac{h_b T}{Q+2} \right\} P\{N(T) = Q+1\} \\ &\quad - \sum_{n=Q+2}^{\infty} \left\{ \frac{\beta T}{n+1} + \frac{h_b T}{n+1} \right\} P\{N(T) = n\}.\end{aligned}$$

Since  $v < b$  we have  $v - b < 0$ . A condition of  $\alpha$  and  $\beta$  is that  $\alpha - \beta T \geq 0$ .

Therefore,  $-\alpha + \beta T \leq 0$ , which in turn implies that

$$-\alpha + \frac{\beta(Q+1)T}{Q+2} \leq 0.$$

As a result,  $\Delta^2\Pi_b(Q) < 0$ , and hence  $\Pi_b(Q)$  is a strictly concave function of  $Q$ . ■

Since  $\Pi_b(Q)$  is a strictly concave function of  $Q$ , it has a unique maximizer of this



function. Let  $Q_d^*$  be the maximizer of  $\Pi_b(Q)$ , then it should satisfy:

$$\Pi_b(Q_d^* + 1) - \Pi_b(Q_d^*) < 0, \quad (7.16)$$

and

$$\Pi_b(Q_d^* - 1) - \Pi_b(Q_d^*) < 0. \quad (7.17)$$

The above system of inequalities is equivalent to saying that  $Q_d^*$  is the smallest  $Q$  satisfying  $\Pi_b(Q + 1) - \Pi_b(Q) < 0$ . That is,

$$Q_d^* = \inf\{Q : \Pi_b(Q + 1) - \Pi_b(Q) < 0, Q \text{ positive int.}\}. \quad (7.18)$$

Now, we reduce Expression (7.15) and provide different forms of it to write the solution as described by the system of inequalities (7.16)–(7.17).

observe that Expression (7.15) can be rewritten as

$$\begin{aligned} \Delta \Pi_b(Q) &= -c + (v - h_b T) P\{N(T) \leq Q\} \\ &+ \sum_{n=Q+1}^{\infty} \left\{ \alpha - \frac{(\beta + h_b)T(Q+1)}{n+1} + b \right\} P\{N(T) = n\} \end{aligned}$$

After substituting

$$P\{N(T) = n\} = \frac{e^{-\lambda T} (\lambda T)^n}{n!}$$

in the above expression, we have

$$\begin{aligned} \Delta \Pi_b(Q) &= -c + (v - h_b T) P\{N(T) \leq Q\} \\ &+ (\alpha + b)(1 - P\{N(T) \leq Q\}) \\ &- \frac{(\beta + h_b)(Q+1)T}{\lambda T} \sum_{n=Q+1}^{\infty} \frac{e^{-\lambda T} (\lambda T)^{(n+1)}}{(n+1)!}, \end{aligned}$$

which leads to

$$\begin{aligned}\Delta\Pi_b(Q) &= -c + \alpha + b + (v - h_b T - \alpha - b)P\{N(T) \leq Q\} \\ &\quad - \frac{(\beta + h_b)(Q + 1)}{\lambda} \sum_{n=Q+2}^{\infty} \frac{e^{-\lambda T} (\lambda T)^n}{n!}.\end{aligned}$$

As a result,

$$\begin{aligned}\Delta\Pi_b(Q) &= -c + \alpha + b + (v - h_b T - \alpha - b)P\{N(T) \leq Q\} \\ &\quad - \frac{(\beta + h_b)(Q + 1)}{\lambda} P\{N(T) \geq Q + 2\}.\end{aligned}\tag{7.19}$$

Observe that Expression (7.19) implies

$$\begin{aligned}\Pi_b(Q - 1) - \Pi_b(Q) &= c - \alpha - b - (v - h_b T - \alpha - b)P\{N(T) \leq (Q - 1)\} \\ &\quad - \frac{(\beta + h_b)Q}{\lambda} P\{N(T) \geq Q + 1\}.\end{aligned}\tag{7.20}$$

Therefore,  $Q_d^*$  should satisfy the following system of inequalities.

$$\begin{aligned}\Pi_b(Q_d^* + 1) - \Pi_b(Q_d^*) &= -c + \alpha + b + (v - h_b T - \alpha - b)P\{N(T) \leq Q_d^*\} \\ &\quad - \frac{(\beta + h_b)(Q_d^* + 1)}{\lambda} P\{N(T) \geq Q_d^* + 2\} < 0,\end{aligned}\tag{7.21}$$

and

$$\begin{aligned}\Pi_b(Q_d^* - 1) - \Pi_b(Q_d^*) &= c - \alpha - b - (v - h_b T - \alpha - b)P\{N(T) \leq (Q_d^* - 1)\} \\ &\quad - \frac{(\beta + h_b)Q_d^*}{\lambda} P\{N(T) \geq Q_d^* + 1\} < 0.\end{aligned}\tag{7.22}$$

Next, we analyze the vendor's subproblem.

### Vendor's Subproblem

In this setting, the vendor simply ships the required number of items to the buyer while incurring a fixed replenishment cost. Thus, for a given value of  $Q$ , the vendor's

profit is

$$\Pi_v(Q) = (c - p)Q - K_v. \quad (7.23)$$

That is, the vendor has no decision variable in its decentralized model.

### VII.1.2. Centralized Model

Let  $\Pi_c(Q)$  be the expected system profits,. That is,  $\Pi_c(Q)$  denotes the sum of the buyer's and the vendor's expected profits, as a function of the buyer's order quantity. Therefore,

$$\Pi_c(Q) = \Pi_b(Q) + \Pi_v(Q). \quad (7.24)$$

Utilizing Expressions (7.14) and (7.23) in the above equation, we obtain

$$\begin{aligned} \Pi_c(Q) = & -cQ - K_b + Q(v - h_b T)P\{N(T) = 0\} + \\ & \sum_{n=1}^Q \left\{ \alpha n - \frac{\beta n T}{2} + \frac{h_b n T}{2} - Q h_b T + (Q - n)v \right\} P\{N(T) = n\} + \\ & \sum_{n=Q+1}^{\infty} \left\{ \alpha Q - \frac{\beta Q(Q+1)T}{2(n+1)} - \frac{h_b Q(Q+1)T}{2(n+1)} - (n - Q)b \right\} P\{N(T) = n\} \\ & + (c - p)Q - K_v. \end{aligned}$$

which can be reduced to

$$\begin{aligned} \Pi_c(Q) = & -pQ - K_b - K_v + Q(v - h_b T)P\{N(T) = 0\} + \\ & \sum_{n=1}^Q \left\{ \alpha n - \frac{\beta n T}{2} + \frac{h_b n T}{2} - Q h_b T + (Q - n)v \right\} P\{N(T) = n\} + \\ & \sum_{n=Q+1}^{\infty} \left\{ \alpha Q - \frac{\beta Q(Q+1)T}{2(n+1)} - \frac{h_b Q(Q+1)T}{2(n+1)} - (n - Q)b \right\} P\{N(T) = n\}. \end{aligned} \quad (7.25)$$

The above equation is essentially the same as (7.14) except that  $c$  and  $K_b$  in (7.14) are replaced by  $p$  and  $K_b + K_v$ , respectively. Therefore, the properties of (7.14) and

(7.25) are similar. In fact, concavity of  $\Pi_c(Q)$  can be proved by replacing  $c$  and  $K_b$  by  $p$  and  $K_b + K_v$ , respectively, in Expressions (7.15) through (7.20). As a result, the optimal value of  $Q$  in the centralized model, denoted by  $Q_c^*$ , should satisfy the following system of inequalities:

$$\begin{aligned} \Pi_c(Q_c^* + 1) - \Pi_c(Q_c^*) &= -p + \alpha + b + (v - h_b T - \alpha - b)P\{N(T) \leq Q_c^*\} \\ &\quad - \frac{(\beta + h_b)(Q_c^* + 1)}{\lambda} P\{N(T) \geq Q_c^* + 2\} < 0, \end{aligned} \quad (7.26)$$

$$\begin{aligned} \Pi_c(Q_c^* - 1) - \Pi_c(Q_c^*) &= p - \alpha - b - (v - h_b T - \alpha - b)P\{N(T) \leq (Q_c^* - 1)\} \\ &\quad - \frac{(\beta + h_b)Q_c^*}{\lambda} P\{N(T) \geq Q_c^* + 1\} < 0. \end{aligned} \quad (7.27)$$

Equivalently,

$$Q_c^* = \inf\{Q : \Pi_c(Q + 1) - \Pi_c(Q) < 0, Q \text{ positive int.}\} \quad (7.28)$$

Observe that (7.26) and (7.27) are similar to (7.21) and (7.22), respectively.

### VII.1.3. Channel Coordination

Before addressing channel coordination issues for the problem of interest, we present the following proposition that compares the optimal solutions of the decentralized and centralized models.

**PROPOSITION 23** In the Buyer-Vendor Problem with Time Dependent Selling Price,  $Q_c^* \geq Q_d^*$ .

**Proof:**

Recalling Expression (7.18), it can be easily shown that  $Q_d^*$  is the smallest value of  $Q$  that satisfies

$$\begin{aligned} & -c + \alpha + b + (v - h_b T - \alpha - b)P\{N(T) \leq Q\} \\ & - \frac{(\beta + h_b)(Q + 1)}{\lambda} P\{N(T) \geq Q + 2\} < 0. \end{aligned} \quad (7.29)$$

Similarly, recalling Expression (7.28), it can be easily shown that  $Q_c^*$  is the smallest value of  $Q$  that satisfies

$$\begin{aligned} & -p + \alpha + b + (v - h_b T - \alpha - b)P\{N(T) \leq Q\} \\ & - \frac{(\beta + h_b)(Q + 1)}{\lambda} P\{N(T) \geq Q + 2\} < 0. \end{aligned} \quad (7.30)$$

Suppose that  $Q_c^* < Q_d^*$ , and let

$$h(Q) = (v - h_b T - \alpha - b)P\{N(T) \leq Q\} - \frac{(\beta + h_b)(Q + 1)}{\lambda} P\{N(T) \geq Q + 2\}.$$

It follows from (7.29) that  $Q_d^*$  is the smallest  $Q$  for which  $-c + \alpha + b + h(Q) < 0$ . This implies that if  $Q_c^* < Q_d^*$ , then  $-c + \alpha + b + h(Q_c^*) \geq 0$ . However, since  $p \leq c$ , we have  $-p + \alpha + b \geq -c + \alpha + b$  so that  $-p + \alpha + b + h(Q_c^*) \geq -c + \alpha + b + h(Q_c^*) \geq 0$  which contradicts (7.30). Hence,  $Q_c^* \geq Q_d^*$ . ■

The above proposition implies that, it is always advantageous for the vendor to receive larger orders from the buyer to achieve channel coordination. An all-unit discount schedule that offers a unit discount for order sizes larger than or equal to  $Q_c^*$ , can be used by the vendor to influence the ordering behavior of the buyer.

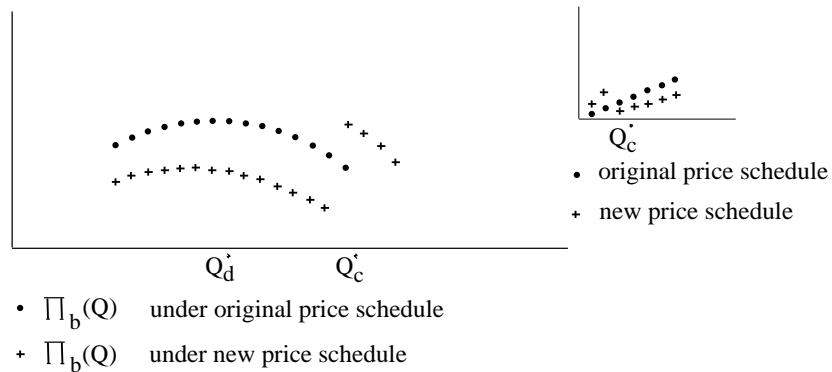
**PROPOSITION 24** Let  $\Delta = \frac{\Pi_b(Q_d^*, c) - \Pi_b(Q_c^*, c)}{Q_c^*}$  and  $c' = c - \Delta$ . Under a unit discount of  $\Delta$  for order sizes greater than or equal to  $Q_c^*$ ,  $Q_c^*$  maximizes the buyer's expected profit function. Furthermore,  $\Pi_b(Q_c^*, c') = \Pi_b(\bar{Q}_d^*, c)$  where  $\Pi_b(Q, c)$  denotes the buyer's expected profits for an order quantity of  $Q$  units under the unit price  $c$ .

**Proof:** The proof of this proposition is the same as that of Proposition 17 in Chapter VI and follows by replacing Expression (7.14) for the buyer's expected profit function.

■

As a result of the above discount scheme the buyer is indifferent between ordering  $Q_d^*$  and  $Q_c^*$ . The vendor can force the buyer to order  $Q_c^*$  by slightly increasing the unit price for order sizes less than  $Q_c^*$ . The resulting price schedule is illustrated in Figure 19.

Figure 19 Proposed Coordination Mechanism



However, as we illustrate next, more efficient coordination mechanisms can be designed if  $T$  is also considered as a negotiable term between the buyer and the vendor. We will illustrate this idea on the following example.

**Example 11** Consider a buyer-vendor system with the following parameter values:  $c = 10$ ,  $p = 6$ ,  $v = 5$ ,  $b = 6$ ,  $h_b = 0.5$ ,  $\lambda = 10$ ,  $\alpha = 20$ ,  $\beta = 2$ ,  $K_b = 20$ ,  $K_v = 25$  and  $T = 1$ .

First row of Table IX compares the objective function values in the decentralized and centralized models for the above example. The last column of Table IX shows the unit discount necessary to coordinate the system using an all-unit discount schedule.

Table IX Profit Comparison Under Different Negotiable Terms

$T$	$Q_d^*$	$Q_c^*$	$\Pi_c(Q_d^*)$	$\Pi_c(Q_c^*)$	$\Pi_b(Q_d^*)$	$\Pi_b(Q_c^*)$	$\Pi_c(Q_c^*) - \Pi_c(Q_d^*)$	Discount
1	12	15	69.087	73.556	46.088	38.556	4.469	0.502
2	22	25	163.825	169.437	100.825	94.437	5.612	0.256
3	31	35	232.822	241.077	133.823	126.077	8.256	0.221
4	40	45	279.946	288.908	144.846	133.908	8.961	0.245
5	48	53	306.8	325.157	139.8	138.157	18.361	0.031
6	55	62	305.602	324.392	110.602	98.713	18.789	0.192
7	61	69	290.337	315.797	71.337	64.8	25.46	0.095
8	63	75	250.28	288.341	23.28	13.34	38.061	0.133

In the other rows of Table IX, the same problem is solved by changing  $T$  in Example 11. We note that the expected profits of the centralized and decentralized models are dependent on  $T$ . Furthermore, the difference of the optimal system profits in the two modeling approaches (i.e.  $\Pi_c(Q_c^*) - \Pi_c(Q_d^*)$ ) increases as  $T$  increases. Therefore, we think that more efficient coordination mechanisms can be designed where  $T$  is also considered as a negotiable term between the buyer and the vendor. In our example, considering both the order quantity and the length of the planning horizon as negotiable terms, the buyer would want to order 40 units and to be in the market for  $T = 4$  units of time. In this case, he/she would maximize his/her own profits (i.e.  $\Pi_b(Q_d^*) = 144.846$ ). However, the system profits would be maximized at  $T = 5$  and  $Q_d^* = 48$ , in which case, the buyer would lose 5.046, but the system would gain 36.242. The cost savings from the centralized solution (i.e., 36.242) can be used to compensate the buyer as well as to increase the expected profits of the parties.

## VII.2. Single Period, Single Replenishment Model with Time Dependent Demand and Retail Price

Now, let us consider the case where the demand arrival process is a Nonhomogenous Poisson Process (NPP). Our goal is to model the case where demand increases in time while retail price decreases. Note that one common demand function with this property in the economics literature (see Mas-Colell et al., page 386), is given by  $a - yr$  where  $r$  is the retail price and  $a$  and  $y$  are positive real constants. Using this relationship between demand and retail price, and noting that we also consider the case  $r(t) = \alpha - \beta t$ , we have

$$a - yr(t) = a - y\alpha + y\beta t = \lambda_0 + \lambda_1 t.$$

Since we assume random demand, we let the rate function be  $\lambda(t) = \lambda_0 + \lambda_1 t$  where  $\lambda_0 \geq 0$  and  $\lambda_1 > 0$ .

Before going into the details of the models, we present some preliminary information about NPPs that will be used later in this chapter.

For an NPP with intensity function  $\lambda(t)$ , if we let

$$m(t) = \int_0^t \lambda(s) ds, \tag{7.31}$$

then,

$$P\{N(t+s) - N(t) = k\} = \frac{e^{-m(t+s)+m(t)} (m(t+s) - m(t))^k}{k!} \tag{7.32}$$

That is,  $N(t+s) - N(t)$  is Poisson distributed with mean  $m(t+s) - m(t)$  (see Ross (1996), pp. 78–79).

**PROPERTY 21** Let  $t \geq 0$  be fixed, and  $U_1, U_2, \dots, U_n$  denote  $n$  i.i.d. random



variables with common distribution

$$P\{U_i \leq u\} = \frac{m(u)}{m(t)} \quad 0 \leq u \leq t. \quad (7.33)$$

Also, let  $\bar{U}_1, \bar{U}_2, \dots, \bar{U}_n$  denote the order statistics of  $U_1, U_2, \dots, U_n$ , and  $S_1, S_2, \dots, S_n$  be the event times in a NPP( $\lambda(\cdot)$ ). Then, given  $N(t) = n$ ,  $(S_1, S_2, \dots, S_n) \stackrel{d}{=} (U_1, U_2, \dots, U_n)$  (see Kulkarni (1995), pp. 227–228).

Now, we are ready to present the decentralized and centralized models for this problem.

### VII.2.1. Decentralized and Centralized Models

The buyer's expected profits in the decentralized model and the total expected system profits can again be computed using conditioning arguments as in Section VII.1. For these calculations we need the expected value of the  $k^{th}$  order statistics of  $U_1, U_2, \dots, U_n$  defined in Property 21 (i.e.,  $E[\bar{U}_k]$ ). However, as we show below, the expression for  $E[\bar{U}_k]$  does not simplify as in the case of Poisson demand arrivals.

#### Calculation of $E[\bar{U}_k]$ :

In order to calculate  $E[\bar{U}_k]$ , we use the density function of the  $k^{th}$  order statistics of  $U_1, U_2, \dots, U_n$ . Recalling Theorem 6, this density function is given by

$$f_{\bar{U}_k}(t) = \frac{n!}{(k-1)!(n-k)!} F_U(t)^{k-1} f_U(t) (1 - F_U(t))^{(n-k)}. \quad (7.34)$$

$F_U(t)$  and  $f_U(t)$  denote the distribution and density functions of  $U_i$  defined in Property 21. Using Expressions (7.31) and (7.33), we have

$$F_U(t) = \frac{m(t)}{m(T)} = \frac{\int_0^t (\lambda_0 + \lambda_1 s) ds}{\int_0^T (\lambda_0 + \lambda_1 s) ds} = \frac{t(\lambda_0 + \lambda_1 t/2)}{T(\lambda_0 + \lambda_1 T/2)}, \quad (7.35)$$

and

$$f_U(t) = \frac{\lambda_0 + \lambda_1 t}{T(\lambda_0 + \lambda_1 T/2)}. \quad (7.36)$$

Substituting Expressions (7.35) and (7.36) in Expression (7.34) leads to

$$f_{\bar{U}_k}(t) = \frac{n!}{(k-1)!(n-k)!} \left[ \frac{t(\lambda_0 + \lambda_1 t/2)}{T(\lambda_0 + \lambda_1 T/2)} \right]^{k-1} \frac{\lambda_0 + \lambda_1 t}{T(\lambda_0 + \lambda_1 T/2)} \left[ 1 - \frac{t(\lambda_0 + \lambda_1 t/2)}{T(\lambda_0 + \lambda_1 T/2)} \right]^{n-k},$$

and hence  $E[\bar{U}_k]$  is given by

$$\frac{n!}{(k-1)!(n-k)!} \int_0^T \left\{ \frac{\lambda_0 + \lambda_1 t}{\lambda_0 + \lambda_1 T/2} \left[ \frac{t(\lambda_0 + \lambda_1 t/2)}{T(\lambda_0 + \lambda_1 T/2)} \right]^k \left[ 1 - \frac{t(\lambda_0 + \lambda_1 t/2)}{T(\lambda_0 + \lambda_1 T/2)} \right]^{n-k} \right\} dt. \quad (7.37)$$

In order to simplify the above integral, we define

$$x = \frac{t(\lambda_0 + \lambda_1 t/2)}{T(\lambda_0 + \lambda_1 T/2)}, \quad (7.38)$$

which leads to

$$\frac{\lambda_1 t^2}{2} + \lambda_0 t - T(\lambda_0 + \lambda_1 T/2)x = 0.$$

The two real roots of the above expression are given by

$$t_{1,2} = \frac{-\lambda_0 \mp \sqrt{\lambda_0^2 + 2\lambda_1 T(\lambda_0 + \lambda_1 T/2)x}}{\lambda_1}.$$

Since we are interested in  $t > 0$ , we should have

$$t = \frac{-\lambda_0 + \sqrt{\lambda_0^2 + 2\lambda_1 T(\lambda_0 + \lambda_1 T/2)x}}{\lambda_1},$$

and hence

$$\lambda_0 + \lambda_1 t/2 = \frac{\lambda_0 + \sqrt{\lambda_0^2 + 2\lambda_1 T(\lambda_0 + \lambda_1 T/2)x}}{2}.$$

From Expression (7.38), we have the following results:

- If  $t = 0$ , then  $x = 0$ ,

- If  $t = T$ , then  $x = 1$ ,
- $(\lambda_0 + \lambda_1 t)dt = T(\lambda_0 + \lambda_1 T/2)dx$ .

Therefore, Expression (7.37) simplifies to

$$E[\bar{U}_k] = \frac{n}{(k-1)!(n-k)!} 2(\lambda_0 + \lambda_1 T/2)T \int_0^1 \frac{x^k(1-x)^k}{\lambda_0 + \sqrt{\lambda_0^2 + 2\lambda_1 T(\lambda_0 + \lambda_1 T/2)}x} dx. \quad (7.39)$$

However, the above integral cannot be computed analytically. Therefore, it should be calculated numerically.

### Expected Revenue of the Buyer:

Note that we can again use Equation (7.3) as a general expression for the buyer's expected revenue. As in the Poisson Process case, we have  $E[\text{Revenue}|N(T) = 0] = 0$ .

For  $1 \leq n \leq Q$ ,

$$E[\text{Revenue}|N(T) = n] = E \left[ \sum_{i=1}^{N(T)} (\alpha - \beta S_i) | N(T) = n \right].$$

From Property 21 and under the independence assumption of  $N(T)$  and  $S_i$ , we have

$$E \left[ \sum_{i=1}^{N(T)} (\alpha - \beta S_i) | N(T) = n \right] = E \left[ \sum_{i=1}^n (\alpha - \beta \bar{U}_i) \right]$$

where  $\bar{U}_i$  is the  $i^{\text{th}}$  order statistics of  $n$  i.i.d. random variables  $U_1, U_2, \dots, U_n$  with common distribution  $P\{U_i \leq u\} = \frac{m(u)}{m(T)}$ . For  $1 \leq n \leq Q$ , we again have

$$E \left[ \sum_{i=1}^{N(T)} (\alpha - \beta S_i) | N(T) = n \right] = \alpha n - \beta E \left[ \sum_{i=1}^n U_i \right] = \alpha n - \beta n E[U_i].$$

Using the density function of  $U_i$  given in Expression (7.36), it can be easily shown that

$$E[U_i] = \frac{(3\lambda_0 + 2\lambda_1 T)T}{(2\lambda_0 + \lambda_1 T)3}.$$

Therefore, for  $1 \leq n \leq Q$

$$E[\text{Revenue}|N(T) = n] = n \left[ \alpha - \beta \frac{(3\lambda_o + 2\lambda_1 T)T}{(2\lambda_o + \lambda_1 T)3} \right].$$

For  $Q < n < \infty$ ,

$$E[\text{Revenue}|N(T) = n] = E \left[ \sum_{i=1}^Q (\alpha - \beta S_i) | N(T) = n \right]$$

Again, using a similar argument, we can write

$$E \left[ \sum_{i=1}^Q (\alpha - \beta S_i) | N(T) = n \right] = \alpha Q - \beta E \left[ \sum_{i=1}^Q \bar{U}_i \right] = \alpha Q - \beta \sum_{i=1}^Q E[\bar{U}_i].$$

Therefore,  $E[\text{Revenue}]$  of the buyer is given by

$$\begin{aligned} E[\text{Revenue}] &= \left[ \alpha - \beta \frac{(3\lambda_o + 2\lambda_1 T)T}{(2\lambda_o + \lambda_1 T)3} \right] \sum_{n=1}^Q n P\{N(T) = n\} \\ &\quad + \sum_{n=Q+1}^{\infty} \left\{ \alpha Q - \beta \sum_{i=1}^Q E[\bar{U}_i] \right\} P\{N(T) = n\} \end{aligned} \quad (7.40)$$

where  $E[\bar{U}_i]$  can be found using Expression (7.39).

### **Expected Inventory Holding Cost of the Buyer:**

The buyer's expected inventory holding cost during the period  $[0, T]$  can again be calculated using Expression (7.7). We analyze the following cases.

For  $N(T) = n = 0$ , we have

$$E[\text{Holding cost}|N(T) = 0] = Qh_b T.$$

For  $1 \leq n \leq Q$ , using a similar argument as in Section VII.1 we have

$$\begin{aligned}
E[\text{Holding cost}|N(T) = n] &= E \left[ \left\{ \sum_{i=1}^{N(T)} S_i h_b + (Q - N(T)) h_b T \right\} | N(T) = n \right] \\
&= E \left[ \sum_{i=1}^n \bar{U}_i h_b + (Q - n) h_b T \right] \\
&= n h_b E[U_i] + (Q - n) h_b T \\
&= n h_b \frac{(3\lambda_o + 2\lambda_1 T) T}{(2\lambda_o + \lambda_1 T) 3} + (Q - n) h_b T \\
&= Q h_b T - \frac{n T h_b (\lambda_1 T + 3\lambda_o)}{3(2\lambda_o + \lambda_1 T)}. \tag{7.41}
\end{aligned}$$

For  $Q < n < \infty$ ,  $E[\text{Holding cost}|N(T) = n]$  should be computed using Expression (7.39) and is given by

$$\begin{aligned}
E[\text{Holding cost}|N(T) = n] &= E \left[ \sum_{i=1}^Q S_i h_b | N(T) = n \right] \\
&= h_b \sum_{i=1}^Q E[\bar{U}_i]. \tag{7.42}
\end{aligned}$$

Using Expressions (7.41) and (7.42) and the fact that  $E[\text{Holding cost}|N(T) = 0] = Q h_b T$ , buyer's expected inventory holding costs can be written as

$$\begin{aligned}
E[\text{Holding cost}] &= Q h_b T P\{N(T) = 0\} + \\
&\quad \sum_{n=1}^Q \left\{ Q h_b T - \frac{n T h_b (3\lambda_o + \lambda_1 T)}{3(2\lambda_o + \lambda_1 T)} \right\} P\{N(T) = n\} + \\
&\quad \sum_{n=Q+1}^{\infty} h_b \sum_{i=1}^Q E[\bar{U}_i] P\{N(T) = n\} \tag{7.43}
\end{aligned}$$

where  $E[\bar{U}_i]$  is given by Expression (7.39).

Buyer's expected profits can be computed using Expression (7.1). The first and third terms of this expression are given by (7.40) and (7.43), respectively. Expected salvage value, expected lost sale cost, and expected sum of purchase and replenishment

cost expressions are again calculated using Expressions (7.11), (7.12) and (7.13). In order to obtain an expression for the expected system profits,  $c$  and  $K_b$  in these expressions should be replaced by  $p$  and  $K_b + K_v$ , respectively.

### VII.2.2. Channel Coordination

Although the analysis for the demand arrival process in the case of NPP does not simplify as does the pure Poisson Process, we can still prove the following proposition:

**PROPOSITION 25** In the Buyer-Vendor Problem with Time Dependent Demand and Retail Price,  $Q_c^* \geq Q_d^*$ .

**Proof:** From Expression (7.23) we have that  $d\Pi_v(Q)/dQ = (c - p) > 0$ . Therefore, vendor's expected profits are increasing in buyer's order quantity. Hence it follows from Proposition 1 that  $Q_c^* \geq Q_d^*$ . ■

The above proposition is important in characterizing the general properties of a mechanism to coordinate the channel. The most efficient coordination mechanism for a problem depends on its characteristics and practical constraints. Again,  $T$  can be treated as a negotiable term between the buyer and the vendor. Since demand is dependent on retail price which in turn depends on time, time dependent incentives can be offered by the vendor to change the ordering behavior of the buyer. For example, a time varying rebate value offered by the vendor to the buyer for each item that is sold, can increase the order quantity of the buyer.

### VII.3. Summary

This chapter studies two single-period stochastic demand problems with decreasing retail price in time. In the first problem, demand is independent of the retail price and in the second problem, demand decreases as retail price increases. The channel

coordination problem in these settings is important because different coordination mechanisms that also take into consideration the length of the planning horizon, can be designed between the parties. For example, in case of consumer electronics or software products, the time until a new version of the product is driven into the market can be a negotiable term. In the second problem, another alternative may be time varying rebate amounts offered by the vendor for each product sold at the buyer. In this case, computing the rebate amount and characterizing its explicit dependency on time, is a research challenge. However, as illustrated in Section VII.2.1, it is not possible to analyze this problem analytically. One way is to study each problem instance by doing the calculations numerically. Another alternative would be to compare different coordination mechanisms using simulation. In this case a dominating coordination mechanism can be determined. An extension of the problems analyzed in this chapter would be the multi-period stochastic demand versions.

## CHAPTER VIII

### SUMMARY AND CONCLUSIONS

This dissertation investigates the buyer-vendor coordination problem with an emphasis on transportation and supply/replenishment issues in the context of recent supply chain initiatives. The goals of the dissertation are to develop a theoretical understanding and a modeling framework for channel coordination under these considerations and to address the question of under what conditions channel coordination works.

As discussed in Chapter II, the early literature in buyer-vendor coordination assumes that the parties fully cooperate and solve their problems using a centralized approach. However, this may not be a practical modeling approach unless the parties are owned by the same company. Therefore, the current trend in the area focuses on a decentralized modeling approach and investigates the mechanisms by which the independently made decisions of the vendor and the buyer can be coordinated (i.e. channel coordination). Some of the coordination mechanisms that are proposed in the literature include quantity discounts and fixed payments. From our analysis of the literature, we conclude that the complexity and design of these coordination mechanisms change with respect to different factors, such as inventory holding cost at the vendor (i.e., whether or not the vendor holds any inventory), demand and retail price dependency, dispatch policy from vendor to buyer, and stochasticity of demand. However, an important practical issue that is generally ignored in the current literature is transportation capacity and costs. It is important to note also that although a few studies do exist for the stochastic demand setting, we believe this area also needs further attention.

Given the above observations, in Chapter III we solve the centralized pure inventory problem with deterministic and constant demand by incorporating transporta-



tion capacity and costs. We believe that the centralized model is important for two reasons. First, it can be used as a direct solution to the inventory replenishment problem in cases of full cooperation between the vendor and buyer (e.g., VMI systems). Secondly, it can be used as a benchmark to improve the decentralized model. Hence, in Chapter IV, we investigate the channel coordination issue for the centralized models described in Chapter III. We show that under conditions that involve transportation capacities and costs, the previously proposed coordination mechanisms are not sufficient for achieving channel coordination. Therefore, we introduce some new coordination mechanisms by which the decentralized model costs can be decreased to equal those of the centralized model.

Chapter V extends our study to consider a finite production rate at the vendor. In this case, the dispatch policy from the vendor to the buyer has an important effect on the system costs. The current literature in this area focuses on developing different dispatch policies to improve the system costs in a centralized model. Solving this model can be very challenging when the structural complexity of the dispatch policy increases. In Chapter V, we first propose a unified model that can be reduced to the different dispatch policies in common use and that takes into account various issues such as transportation capacities and costs, different pricing strategies, etc. We then illustrate that the algorithms we propose in Chapter III, can be the basis for solving centralized models that include transportation costs and capacities in finite production rate problems. Finally, we compare the different dispatch policies under their original modeling assumptions, for an extensive set of problem instances.

Chapter VI models the transportation capacities and costs in the single period stochastic demand problem. Centralized and decentralized models are developed; their solutions are compared; and some efficient mechanisms are proposed for channel coordination. As a result of our analysis in this and earlier chapters, we conclude that

channel coordination becomes more important when demand is stochastic.

Finally, in Chapter VII, we consider another practical situation where the retail price of items decreases over time. We show that this again is a case where the existing coordination mechanisms are insufficient. We illustrate that an efficient coordination mechanism should take the length of the planning horizon into account.

We believe that apart from its practical contributions, this dissertation makes several theoretical contributions in modeling and algorithmic development. To support this claim, we cite Chapters III, IV, and VI for contributions in algorithmic development and Chapters V and VII for contributions in modeling.

An important generalization of the problems considered within this dissertation is the case of multiple decision makers. When more than one entity is assumed in either the upper or lower echelon, the level of lateral competition becomes another significant factor. In such cases, not only do the centralized models become more challenging, but more sophisticated analytical tools such as game theory become necessary in order to reach efficient solutions.

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